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**Towards a
stable and fast
dynamic
skeletal muscle
model**

Motivation

Forward simulations of three-dimensional, continuum-mechanical skeletal muscle models are computationally demanding and expensive. To adequately represent the muscles' mechanical behaviour, a fully dynamic modelling framework based on the theory of finite hyperelasticity, which accounts for the highly nonlinear, anisotropic, viscoelastic and incompressible material behaviour, needs to be established. Discretisation of the governing equations yields a nonlinear second-order differential algebraic equation system, which represents a challenge for solution strategies as well as for the application of model order reduction techniques.

Governing equations

Let $\Omega_0 \subset \mathbb{R}^3$ be the muscle domain in the reference configuration. The equations describing the muscle deformation over time are the **balance of linear momentum** subject to the **incompressibility constraint**

$$\rho_0(\mathbf{X}) \frac{\partial \mathbf{V}}{\partial t}(\mathbf{X}, t) = \nabla \cdot \mathbf{P}(\mathbf{X}, t) + \mathbf{B}(\mathbf{X}, t), \text{ s.t. } J(\mathbf{X}, t) = 1, \forall \mathbf{X} \in \Omega_0, \quad (1)$$

and a **nonlinear constitutive equation** of the form

$$\mathbf{P}(\mathbf{X}, t) = \mathbf{P}^{iso}(\mathbf{X}, t) + \mathbf{P}^{aniso}(\mathbf{X}, t) + \mathbf{P}^{act}(\mathbf{X}, t) + \mathbf{P}^{visc}(\mathbf{X}, t) - p(\mathbf{X}, t)J(\mathbf{X}, t)\mathbf{F}^{T-1}(\mathbf{X}, t). \quad (2)$$

Herein, ρ_0 is the muscle density, $\mathbf{V}$ is the velocity field, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\mathbf{B}$ are the body forces, $J := \det \mathbf{F}$ is the Jacobian, $\mathbf{F}$ is the deformation gradient and p is the hydrostatic pressure.

Inserting equation (2) into equation (1), introducing a parameter $\mu \in \mathcal{P} \subset \mathbb{R}^p$ for varying muscle properties and discretising the resulting equation using the **finite element method** (Taylor-Hood elements), one obtains the following **second-order DAE system**

$$\mathbf{M}\mathbf{u}''(\mu, t) = -\mathbf{D}(\mu)\mathbf{u}'(\mu, t) - \mathbf{K}(\mathbf{u}(\mu, t), \mu, t) + \mathbf{A}(\mathbf{u}(\mu, t))\mathbf{w}(\mu, t), \quad (3) \\ \text{s.t. } \mathbf{0} = \mathbf{G}(\mathbf{u}(\mu, t)),$$

where $\mathbf{u} \in \mathbb{R}^{3N}$ is the vector of position coefficients, $\mathbf{w} \in \mathbb{R}^{N_p}$ contains the pressure coefficients and $\mathbf{M}, \mathbf{D} \in \mathbb{R}^{3N \times 3N}$ are the mass and viscous damping matrix, respectively. Further, $\mathbf{K} : \mathbb{R}^{3N} \rightarrow \mathbb{R}^{3N}$ is the generalised stiffness operator for the nonlinear deviatoric extra stress, and $\mathbf{A} : \mathbb{R}^{3N} \rightarrow \mathbb{R}^{3N \times N_p}$ contains the volumetric part of $\mathbf{P}$. $\mathbf{G} : \mathbb{R}^{3N} \rightarrow \mathbb{R}^{N_p}$ is the operator associated with the incompressibility constraint, where it holds that $\frac{\partial \mathbf{G}(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{A}(\mathbf{u})^T$.

Methods

Full model

So far, the full second-order DAE system (3) is solved by transforming it into a first-order system introducing the velocity $\mathbf{v}(t) := \mathbf{u}'(t)$,

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}' = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} \mathbf{v} \\ -\mathbf{K}(\mathbf{u}) + \mathbf{A}(\mathbf{u})\mathbf{w} \\ \mathbf{G}(\mathbf{u}) \end{pmatrix}, \quad (4)$$

and by applying a numerical time integration of BDF-type (linear multistep method, MATLAB solver ode15i).

An alternative approach is to perform an **index-reduction** by taking the second derivative of the constraint equation and thereby transforming the index-3-DAE into an index-1-DAE. The resulting system has the form

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & -\mathbf{A}(\mathbf{u}) \\ \mathbf{0} & \mathbf{A}(\mathbf{u})^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}' \\ \mathbf{v}' \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} \mathbf{v} \\ -\mathbf{K}(\mathbf{u}) \\ \varphi(\mathbf{u}, \mathbf{v}) \end{pmatrix}, \quad (5)$$

with $\varphi(\mathbf{u}, \mathbf{v}) := -\frac{\partial^2 \mathbf{G}(\mathbf{u})}{\partial \mathbf{u}^2} \langle \mathbf{v}, \mathbf{v} \rangle$, or making use of the Baumgarte stabilisation $\varphi(\mathbf{u}, \mathbf{v}) := -\frac{\partial^2 \mathbf{G}(\mathbf{u})}{\partial \mathbf{u}^2} \langle \mathbf{v}, \mathbf{v} \rangle + 2\alpha \frac{\partial \mathbf{G}(\mathbf{u})}{\partial \mathbf{u}} \mathbf{v} + \beta^2 \mathbf{G}(\mathbf{u})$.

Reduced model

To obtain the reduced system, we apply a **(Petrov-) Galerkin block-projection** (c.f. [1],[4]) of the first-order system (4) by setting $\mathbf{u} \approx \mathbf{V}_u \mathbf{z}_u$, $\mathbf{v} \approx \mathbf{V}_v \mathbf{z}_v$ and $\mathbf{w} \approx \mathbf{V}_w \mathbf{z}_w$, i.e. by separately projecting the position, velocity and pressure DOFs. Subsequently, the test bases $\mathbf{W}_\alpha \neq \mathbf{V}_\alpha$ for $\alpha \in \{u, v, w\}$ will be chosen.

This yields the reduced system

$$\begin{pmatrix} \mathbf{W}_u^T \mathbf{V}_u & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_v^T \mathbf{M} \mathbf{V}_v & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{z}_u \\ \mathbf{z}_v \\ \mathbf{z}_w \end{pmatrix}' = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_v^T \mathbf{D} \mathbf{V}_v & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{z}_u \\ \mathbf{z}_v \\ \mathbf{z}_w \end{pmatrix} + \begin{pmatrix} \mathbf{W}_u^T \mathbf{V}_v \mathbf{z}_v \\ -\mathbf{W}_v^T \mathbf{K}(\mathbf{V}_u \mathbf{z}_u) + \mathbf{W}_v^T \mathbf{A}(\mathbf{V}_u \mathbf{z}_u) \mathbf{V}_w \mathbf{z}_w \\ \mathbf{W}_w^T \mathbf{G}(\mathbf{V}_u \mathbf{z}_u) \end{pmatrix}.$$

While the reduced bases $\mathbf{V}_\alpha, \alpha \in \{u, v, w\}$, are obtained during the offline phase via a **proper orthogonal decomposition (POD)** on the respective DOFs of the precomputed training data, the on-line evaluation of the projected nonlinear operator can be approximated e.g. by the **discrete empirical interpolation method (DEIM)** [$\mathbf{W}_v^T \mathbf{K}(\mathbf{V}_u \mathbf{z}_u) \approx \mathbf{W}_v^T \mathbf{U}(\mathbf{P}^T \mathbf{U})^{-1} \mathbf{P}^T \mathbf{K}(\mathbf{V}_u \mathbf{z}_u)$], c.f. [2], or the **energy conserving sampling and weighting (ECSW)** hyperreduction method [$\mathbf{W}_v^T \mathbf{K}(\mathbf{V}_u \mathbf{z}_u) \approx \sum_{e \in \tilde{E}} \xi_e^* \mathbf{W}_v^{eT} \mathbf{K}^e(\mathbf{V}_u^e \mathbf{z}_u^e)$], c.f. [3].

Results

To test the performance of the proposed method, an idealised rectangular muscle is considered. The muscle is fixed at its left end while a force of 10 MPa is applied linearly within 30 ms to its right end. Subsequently, a POD is performed on that single trajectory.

Figure 1: The results from the offline phase. (a) The experimental setup: 128 elements, 4408 u -DOFs, 225 w -DOFs, 9041 DOFs in total. (b) The output of interest of the precomputed trajectory: x-position of the right face. (c) The singular value decays of the PODs on the u - and v -DOFs. (d) The singular value decay of the POD on the w -DOFs.

In order to assess the quality of the projection spaces, the reduced state $\mathbf{z} = (\mathbf{z}_u, \mathbf{z}_v, \mathbf{z}_w)^T$ is constructed from the full state $\mathbf{x} = (\mathbf{u}, \mathbf{v}, \mathbf{w})^T$ by multiplication $\mathbf{z} = \mathbf{V}^T \mathbf{x}$, with $\mathbf{V} := \text{diag}(\mathbf{V}_u, \mathbf{V}_v, \mathbf{V}_w)$.

Figure 2: (a) Full state $\mathbf{x} = (\mathbf{u}, \mathbf{v}, \mathbf{w})^T$. (b) Reduced state $\mathbf{z} = (\mathbf{z}_u, \mathbf{z}_v, \mathbf{z}_w)^T$ with $r_u = 130$, $r_v = 140$, $r_w = 90$. (c) Reduced state $\mathbf{z} = (\mathbf{z}_u, \mathbf{z}_v, \mathbf{z}_w)^T$ with $r_u = 13$, $r_v = 14$, $r_w = 9$.

The approximation error $\|\mathbf{x} - \mathbf{V}\mathbf{z}\|_2 = \|\mathbf{x} - \mathbf{V}\mathbf{V}^T \mathbf{x}\|_2$ for the sizes suggested by the singular value decays (c.f. Fig. 2b) is of order 10^{-12} , i.e. the projection spaces represent the full state spaces well enough.

References

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