Advances in Hierarchical Model Reduction and combination with other computational reduction methods

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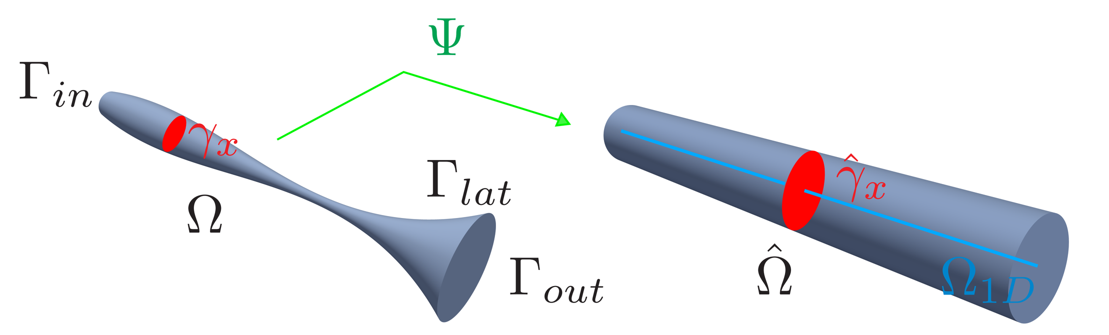
Introduction

When dealing with Parametrized Partial Differential Equations the computational cost required by a large number of solutions for each new value of the involved parameters may be unaffordably large. To mitigate that, different methods have been studied in order to find solutions in a more efficient way. We combine the Hierarchical Model Reduction (HiMod) technique [5, 4, 1], developed for non-parametric equations over domains with a dominant flow direction, with a Reduced Basis method, to efficiently tackle parameter dependence [3]. Depending on the construction of the reduced basis space we end up with two reduced order techniques called HiRB (based on a Greedy algorithm [6]) and HiPOD (based on Proper Orthogonal Decomposition [2]). We present results related to saddle point problems, in particular to Stokes equations, and Optimal Control problems.

1) Hierarchical Model Reduction (HiMod)



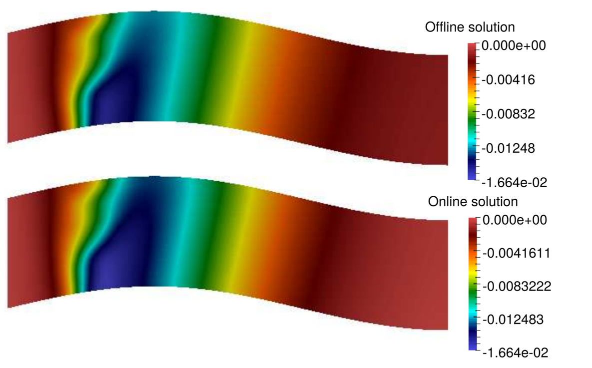
HiMod is based on the decomposition of the domain into a **dominant flow direction** Ω_{1D} and a **transverse** one, γ_x.

We define a discrete **Finite Element** space V_{1D}^h over Ω_{1D}, while the transverse direction is reconstructed by a **modal basis** so that we seek solutions of the form:

$$v_m(x,y) = \sum_{i=1}^{m} \tilde{v}_k(x)\varphi_k(y), \quad \tilde{v}_k(x) \in V_{1D}^h, x \in \Omega_{1D}, y \in \gamma_x.$$

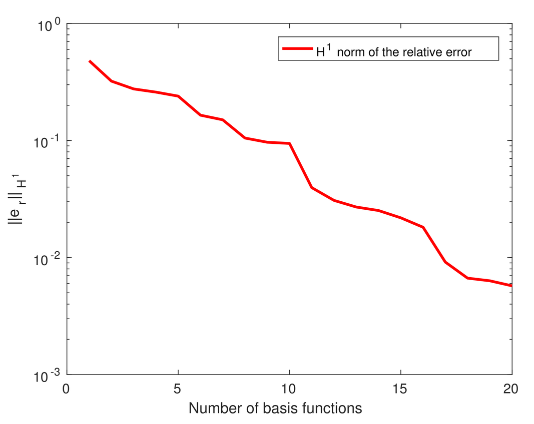
Modal basis functions φ_k are obtained as **eigenfunctions** of a **Sturm-Liouville** problem.

3) HiRB for ADR problems



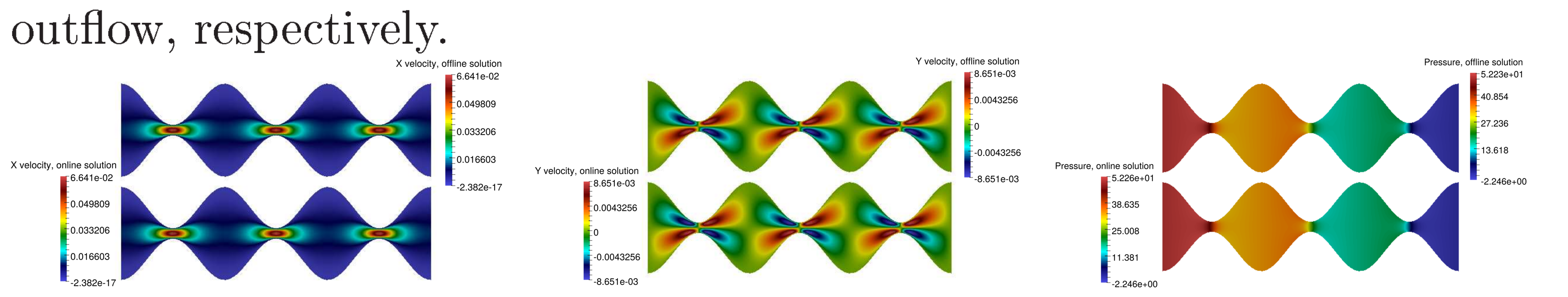
These results are related to **offline** and **online** solutions for an **Advection-Diffusion-Reaction** PDE, where the parameters are allowed to vary on very large ranges. The online solution manifold is composed by 20 HiMod basis functions.

The trend of the H^1 norm of the error is shown on the right. A very good accuracy is reached with a low number of basis functions. Thanks to the dimension reduction, the ratio between offline (HiMod) and online (HiRB) computational times (speedup) is of the order of 10^3.

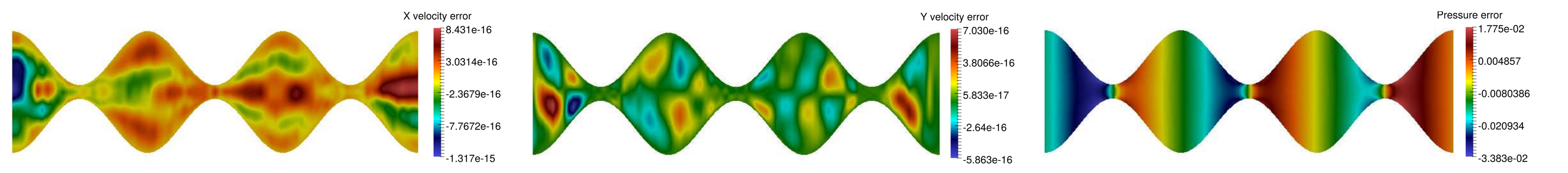


4) HiPOD and HiRB for Stokes equations

Consider the solution to **Stokes equations** in a sinusoidal pipe with non-homogeneous and homogeneous Neumann conditions on the inflow and outflow, respectively.



Online (HiRB) and offline (HiMod) solutions are in very good agreement, i.e., the method is very accurate. These results are obtained with just 3 basis functions of the HiMod solution manifold.



2) HiPOD and HiRB

We use HiMod as the **high fidelity** method employed for projection-based reduction procedures. A faster offline stage is obtained replacing FE discretizations with HiMod computations. We study two different methods:

- **HiPOD**: the reduced space is obtained applying a **Proper Orthogonal Decomposition** to a training set of HiMod solutions;
- **HiRB**: the reduced space is now enlarged step by step with the HiMod solution related to the parameter value that maximizes the error, i.e., by a **Greedy** algorithm.

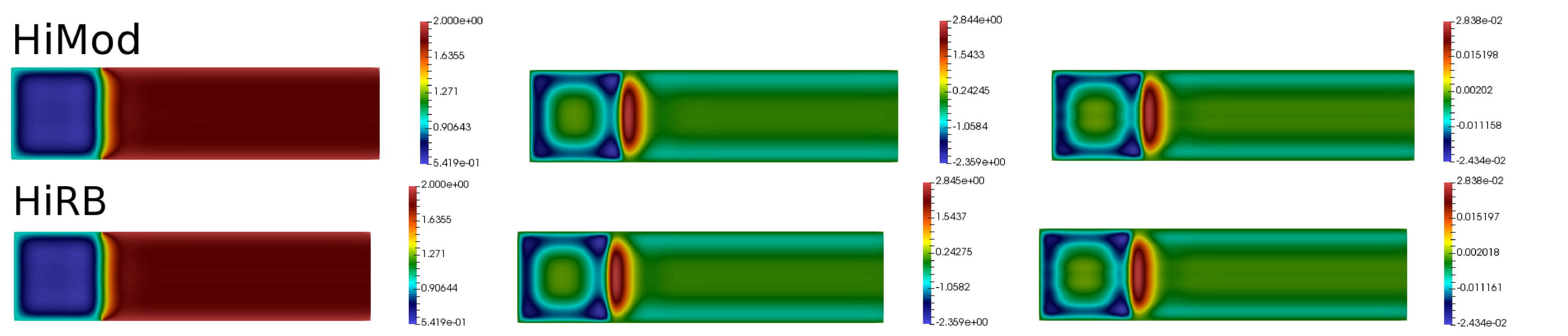
5) HiPOD and HiRB for OCPs

The truth HiMod approximation for an OCP in its saddle-point formulation is obtained defining three uniform hierarchical reduced spaces for the state, control, adjoint variables. We reduce the problem thanks to the POD or the Greedy algorithm as: find $(\mathbf{x}_N^{m,n}(\boldsymbol{\mu}), p_N^m(\boldsymbol{\mu})) \in \mathbf{X}_N^{m,n} \times Q_N^m$ such that

$$\begin{cases} \mathcal{A}(\mathbf{x}_N^{m,n}, \mathbf{y}_N^{m,n}; \boldsymbol{\mu}) + \mathcal{B}(\mathbf{y}_N^{m,n}, p_N^m; \boldsymbol{\mu}) = \langle \mathbf{F}(\boldsymbol{\mu}), \mathbf{y}_N^{m,n} \rangle & \forall \mathbf{y}_N^{m,n} \in \mathbf{X}_N^{m,n}, \\ \mathcal{B}(\mathbf{x}_N^{m,n}, q_N^m; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q_N^m \rangle & \forall q_N^m \in Q_N^m. \end{cases}$$

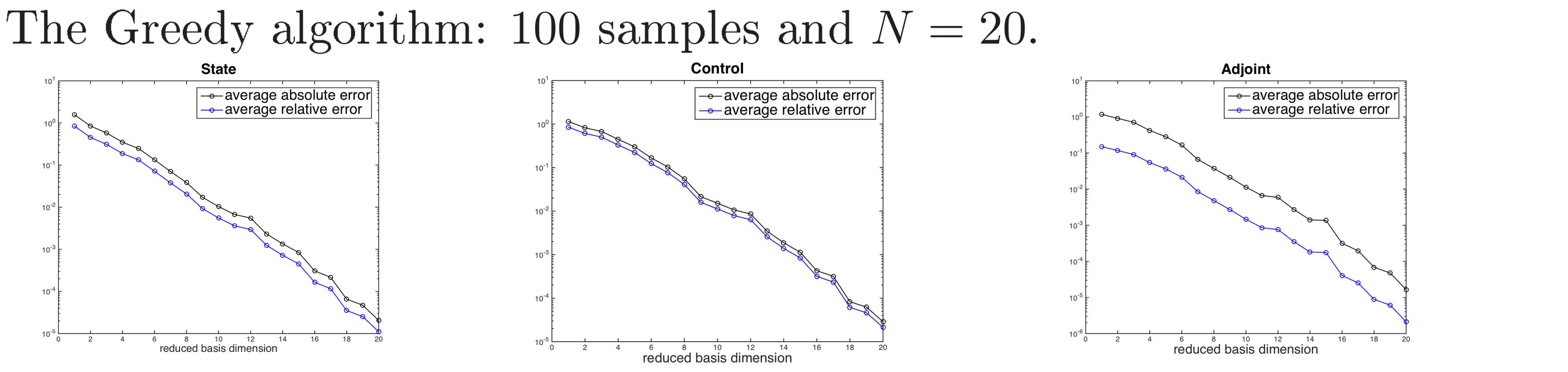
6) HiRB for a linear-elliptic OCP

We consider the online and offline solutions for a parametrized elliptic distributed **optimal control problem**, where the state equation is a Graetz flow. The domain is a rectangle where we impose a non-homogeneous Dirichlet boundary condition on the inlet and along lateral boundaries. We use the Legendre polynomials as modal basis functions for the control variable, and the solutions of a Sturm-Liouville problem as modal basis functions for the state and adjoint variables.



7) HiRB accuracy for OCP problems

The Greedy algorithm: 100 samples and $N = 20$.



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Acknowledgements

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