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Shape Finding in Structural Optimization with Parametrically Reduced Finite Element Models

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Motivation

- weight reduction of technical products can be achieved by optimizing the shape and thereby the mechanical properties of the structure
- conducting optimizations with parameterized, reduced order models instead of parameterized, full order models would be desirable
- commercial, linear finite element codes provide the equation of motion of elastic structures

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{B}\mathbf{u} \qquad (1)$$

only for a **fixed finite element mesh** and for a **fixed geometry**

⇒ the parameter dependency is not retained in the model
⇒ application of PMOR in shape optimizations is challenging
⇒ **a geometrically parameterized solid finite element is derived**
⇒ **the parameterized element is formulated with respect to global design parameters, e.g., spline parameters, to obtain parameterized models which can be efficiently combined with PMOR in a reduced order shape optimization**

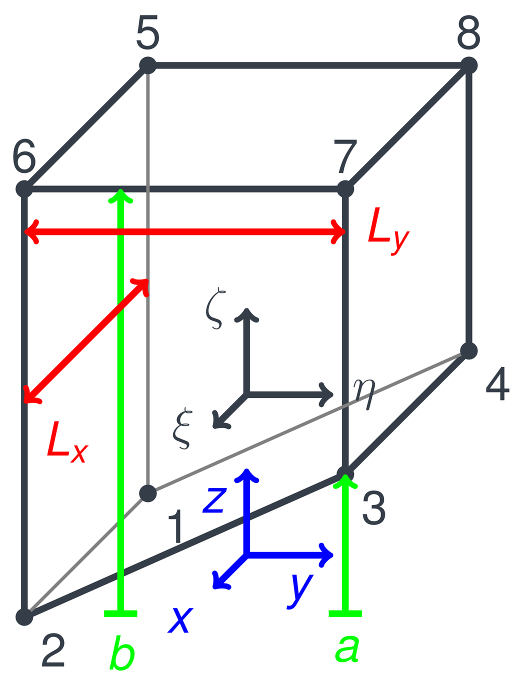
Derivation of a Parameterized Equation of Motion

- solid finite element formulation as in [1]

⇒ displacement field and strain-displacement relation approximated as $\mathbf{d} \approx \mathbf{d}_h = \mathbf{N}\mathbf{q}$ and $\epsilon \approx \epsilon_h = \overline{\mathbf{B}}\mathbf{q}$

- **idea** is to include element geometry parameters in finite element formulation

⇒ element geometry is now parameterized with two parameters $\mathbf{p} = [a,\ b]$
⇒ nodal coordinates of, e.g., node 3 $[\frac{1}{2}L_x, \frac{1}{2}L_y, a]$
⇒ ansatz function no. 3
$N_3(x, y, z, \mathbf{p}) = \frac{1}{8}(x + \frac{1}{2}L_x)(y + \frac{1}{2}L_y)(z - a)$



- parameterized element stiffness matrix becomes

$$\mathbf{k}_e(\mathbf{p}) = \int_\Omega \overline{\mathbf{B}}^T(\mathbf{p})\mathbf{D}\overline{\mathbf{B}}(\mathbf{p})\,\mathrm{d}\Omega = \sum_{j=1}^{J} w_{e,j}(\mathbf{p})\mathbf{k}_{e,j} \qquad (2)$$

- assembly of E parameterized finite elements with distribution matrices $\mathbf{C}_e$ to derive parameterized global stiffness matrix in affine representation

$$\mathbf{K}(\mathbf{p}) = \sum_{e=1}^{E} \mathbf{C}_e \left(\sum_{j=1}^{J} w_{e,j}(\mathbf{p})\mathbf{k}_{e,j} \right) \mathbf{C}_e^T = \sum_{l=1}^{L} w_l(\mathbf{p}) \underbrace{\mathbf{C}_l \mathbf{k}_l \mathbf{C}_l^T}_{=:\mathbf{K}_l} \qquad (3)$$

- parameterized equation of motion

$$\underbrace{\left(\sum_{m=1}^{M} w_m(\mathbf{p})\mathbf{M}_m \right)}_{\mathbf{M}(\mathbf{p})} \ddot{\mathbf{q}} + \underbrace{\left(\sum_{l=1}^{L} w_l(\mathbf{p})\mathbf{K}_l \right)}_{\mathbf{K}(\mathbf{p})} \mathbf{q} = \mathbf{B}\mathbf{u} \qquad (4)$$

- parameterized transfer function

$$\mathbf{H}(s, \mathbf{p}) = \mathbf{C}\left(s^2\mathbf{M}(\mathbf{p}) + \mathbf{K}(\mathbf{p})\right)^{-1}\mathbf{B} \qquad (5)$$

Parametric Model Order Reduction (PMOR)

- approximation of solution $\mathbf{q} \approx \mathbf{V}\mathbf{q}_{\mathrm{red}}$ in lower dimensional subspace
- PMOR with interpolatory methods, see [2]
- matching of transfer function of parameterized, full order and parameterized, reduced order model at expansion point σ_i and $\mathbf{p}_i$

$$\begin{gathered} \mathrm{span}(\mathbf{V}) = \mathrm{span}\left(\left(\sigma_i^2\mathbf{M}(\mathbf{p}_i) + \mathbf{K}(\mathbf{p}_i)\right)^{-1}\mathbf{B}(\mathbf{p}_i) \right) \\ \Rightarrow \mathbf{H}(\sigma_i, \mathbf{p}_i) = \mathbf{H}_{\mathrm{red}}(\sigma_i, \mathbf{p}_i) \end{gathered} \qquad (6)$$

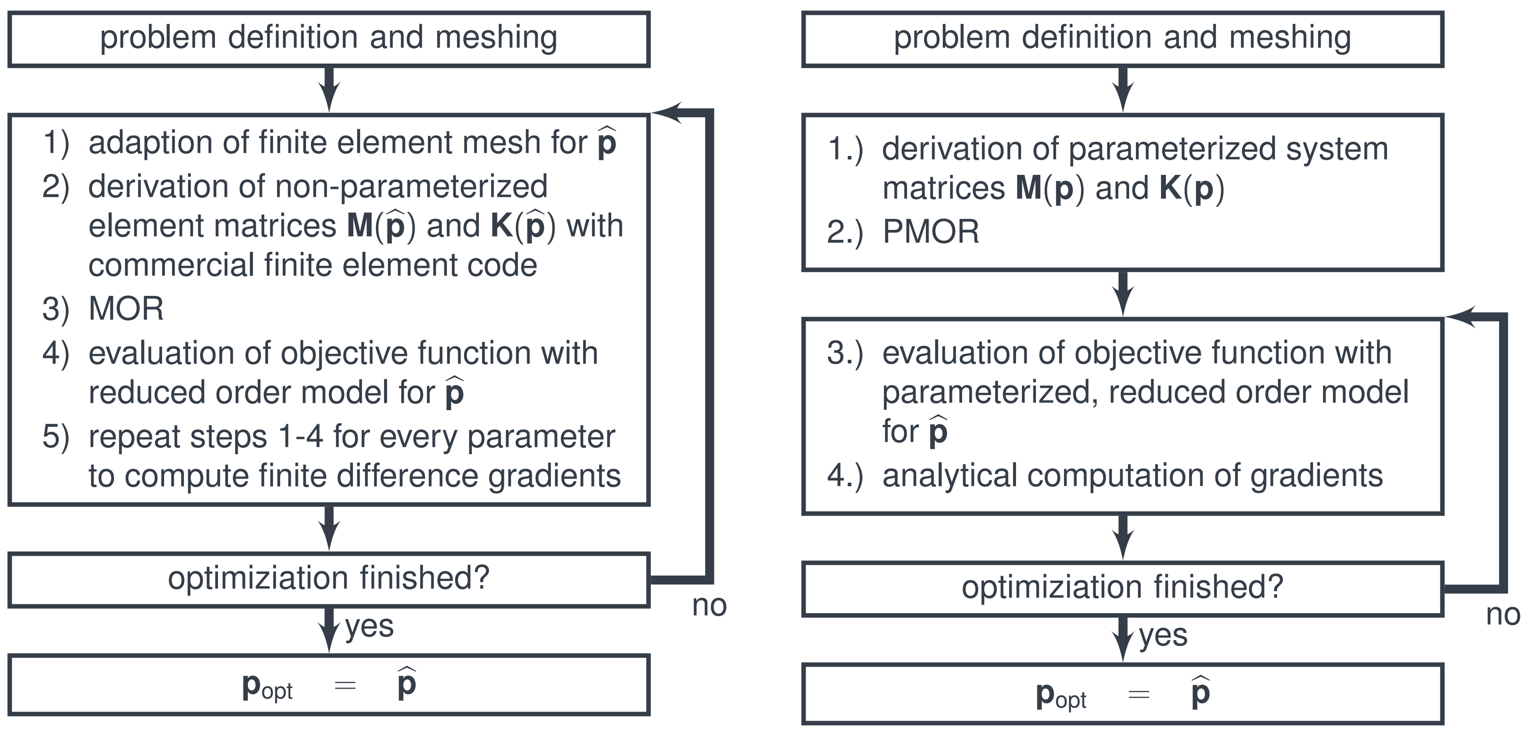
- parameterized, reduced system matrices, e.g., stiffness matrix

$$\mathbf{K}_{\mathrm{red}}(\mathbf{p}) = \mathbf{V}^T \left(\sum_{l=1}^{L} w_l(\mathbf{p})\mathbf{K}_l \right) \mathbf{V} = \sum_{l=1}^{L} w_l(\mathbf{p})\mathbf{V}^T\mathbf{K}_l\mathbf{V} = \sum_{l=1}^{L} w_l(\mathbf{p})\mathbf{K}_{\mathrm{red},l} \qquad (7)$$

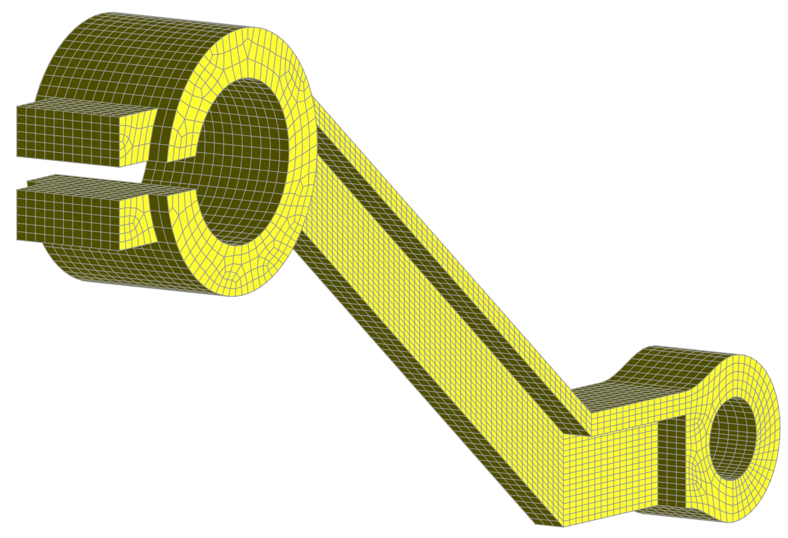
- additional match of gradient of transfer function possible
⇒ **advantageous in an optimization context**

Numerical Shape Optimization Example

- **advantages with proposed approach in an optimization context**
  ❶ no further finite element preprocessing necessary after reduction
  ❷ analytical gradient evaluation possible
- conventional shape optimization (left) and optimization with proposed parameterized element formulation (right)

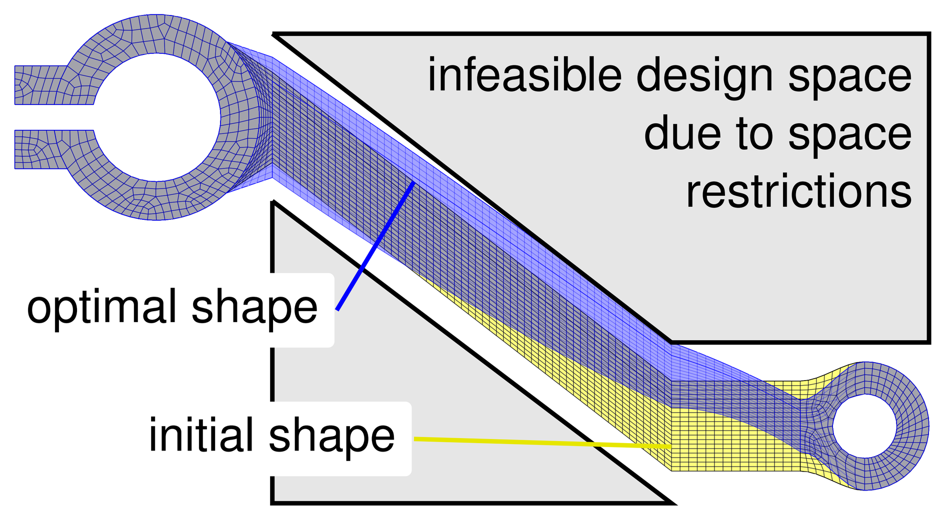


- shape optimization of a support
- nodes at large bore hole pinned
- small bore hole vertically loaded, displacements measured at small bore hole, SISO system
- shape of support parameterized with Bézier curves
- full order model with $N = 50\,784$ degrees of freedom
- objective is to minimize the flexibility in a given frequency band



$$\min \sum_h |\mathbf{H}(s_h, \mathbf{p})|, \quad s_h = h(2\pi \mathrm{i}), \quad h = 0, 1, \ldots, 300$$
$$\text{with} \quad \mathcal{P} := \{\mathbf{p} \in \mathbb{R}^8 \,|\, m(\mathbf{p}) \leq m(\mathbf{p}_0) \,|\, \mathbf{p}_{\mathrm{lower}} \leq \mathbf{p} \leq \mathbf{p}_{\mathrm{upper}}\} \qquad (8)$$

- both optimization approaches deliver the same optimal shape
- **significant speedup** with proposed parameterization compared to conventional non-parameterized approach



	n_{red}	offline	online	total
conventional approach	7	0 s	5862 s	5862 s
proposed approach	64	1248 s	39 s	1287 s

Acknowledgements

The authors gratefully thank the German Research Foundation (DFG) for the support of this research work within the collaborative research centre SFB/CRC 1244 "Adaptive Skins and Structures for the Built Environment of Tomorrow" with the project B01.

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[2] U. Baur, C. Beattie, P. Benner, and S. Gugercin.
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