Article title: An improved bound on the sum of prime numbers

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AN IMPROVED BOUND ON THE SUM OF PRIME NUMBERS.

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ABSTRACT.

We derive two asymptotic formulae for the upper bound on the sum of the first n primes. Both the Supremum and the Estimate of the sum are superior to known bounds. The Estimate bound had been derived to promote efficiency of estimation of the sum.

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M. U. Feliksiak

1. THE SUM OF PRIME NUMBERS: THE R. MANDL'S INEQUALITY.

FIGURE 1. The drawing on the left compares the graphs of Dusart's (in blue), Bach & Shallit (in purple) and Hassani's (in red) estimates as compared to the true sum of primes (in black). The drawing on the right compares the estimation error made by the respective estimates.

FIGURE 2. The drawing compares the estimation error made by the Hassani's estimate (in red) and the absolute error made by the Sharp Estimate S.E. (in blue)

The conjectured inequality by R. Mandl, has been proven by J.B. Rosser and L. Schoenfeld in their 1975 paper [8]. The Mandl's inequality asserts that:

$$(1.1) \quad \sum_{i=1}^n p_i < \frac{n \times p_n}{2}$$

The efforts directed to refine and sharpen the Mandl's estimate include for example Dusart [2] who proved that:

$$(1.2) \quad \sum_{i=1}^n p_i \leq \frac{n^2}{2} \left(\log n + \log \log n - \frac{3}{2} + \frac{\log \log n - 2.29}{\log n} \right) \quad \forall n \geq 10134$$

M. Hassani [7] in 2006 presented a considerably improved refinement:

$$(1.3) \quad \sum_{i=1}^n p_i < \frac{n \times p_n}{2} - \frac{n^2}{14} \quad \forall n \geq 10$$

These estimates however, diverge relatively quickly from the actual sum of the prime numbers. In contrast to that, theory based on the Gauss' offset integral provides for a significantly improved bounds. Let's therefore define:

Definition 1.1 (True sum of prime numbers).

$$\mathcal{T.S.} = \sum_{i=1}^n p_i$$

Definition 1.2 (Supremum Upper Integration Limit).

$$\theta_{(s)} = 2 \sum_{i=1}^n (p_i \log p_i)$$

Definition 1.3 (Sharp Estimate $\mathcal{S.E.}_{(p_n)}$ Upper Integration Limit).

$$\theta_{(e)} = \left(p_n - (\sqrt{5} - 1) (4\gamma^2 - 2\gamma) (\log p_{(n)}) \sqrt[3]{p_{(n)}} \right)^2$$

and $\gamma \approx 0.57721566490153286$ is the Euler-Mascheroni constant.

Definition 1.4 (Absolute Estimation Error of the $\mathcal{S.E.}_{(p_n)}$).

$$\mathcal{E.E.}_{(p_n)} = \left| \left(\mathcal{S.E.}_{(p_n)} = \int_2^{\theta_{(e)}} \frac{dt}{\log t} \right) - \sum_{i=1}^n p_i = \mathcal{T.S.} \right| \quad \forall p_n \in \mathbb{N} \mid p_{(n)} \geq 2$$

In this section we present the Supremum $\mathcal{SUP}_{(p_n)}$ upper bound, as well as the Sharp Estimate $\mathcal{S.E.}_{(p_n)}$. Both these functions depend on the theory of the primorial function and its bounds, presented here:

Lemma 1.5 (Upper Bound on the logarithm of the primorial).

The natural logarithm of the primorial function is strictly less than the respective prime number $p \in \mathbb{N}$:

$$(1.4) \quad \log p_{(n)} \# < p_{(n)} \leq n \quad \forall n \in \mathbb{N} \mid n \geq 2, \text{ where } p_n \text{ is the largest prime } p \leq n$$

In particular the natural logarithm of the primorial function is asymptotic (from below) to the respective prime number:

$$(1.5) \quad \log p_{(n)} \# \sim p_{(n)}$$

Lemma 1.6 (Lower Estimation Error Bound On The Difference $p_n - \log p_n \#$).

The error of estimation of the primorial function by the use of the value of $p_{(n)}$ imposes the following lower bound:

$$(1.6) \quad \mathcal{LB}_{p_{(n)}} = \left(\sqrt{5} - 1 \right) (4\gamma^2 - 2\gamma) (\log p_{(n)}) \sqrt[3]{p_{(n)}} < (p_{(n)} - \log p_{(n)} \#)$$

$$\forall p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2$$

where $\gamma \approx 0.577215664901532860651209$ is the Euler-Mascheroni constant.

Lemma 1.7 (Upper Estimation Error Bound On The Difference $p_n - \log p_n \#$).

The error of estimation of the primorial function by the use of the value of $p_{(n)}$ imposes the following upper bound:

$$(1.7) \quad (p_{(n)} - \log p_{(n)} \#) < 2\sqrt{p_{(n)}} = \mathcal{UB}_{p_{(n)}} \quad \forall p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2$$

For the proofs of Lemmas 1.5, 1.6 and 1.7, please consult Feliksiak [5].

1.1. Supremum Bound on the Sum of Prime Numbers.

The Supremum bound and the Sharp Estimate, both implement a variant of the Gauss' offset logarithmic integral. The Supremum is a very accurate bound, its computation however will become challenging very quickly. The degree of computational difficulty of the Supremum is about the same as computing the sum itself. Consequently, the Sharp Estimate $\mathcal{S.E.}_{(p_n)}$ is so to say, the “golden mean” to achieve the goal.

The Sharp Estimate $\mathcal{S.E.}_{(p_n)}$ is quick and straightforward to compute. Thus, it attenuates the computational difficulty, while at the same time it is sufficiently accurate to rely on. The $\mathcal{S.E.}_{(p_n)}$ cuts through the meanders of the True Sum of the prime numbers, producing alternatively positive/negative estimation error values. By taking the Absolute value of the Estimation Error, we can demonstrate that the estimation error, at the prime $p_n = 19373$ already falls below 1 percent of the value that the True Sum attains. The estimation error improves as $p_n \rightarrow \infty$. The Figures 3 and 4 serve as a visual aid and are not a part of the proofs.

FIGURE 3. The drawings show the graphs of the True Sum of primes (in black), the Supremum bound (in red) and the Sharp Estimate **S.E.** (in blue). Figures are drawn w.r.t. p_n .

FIGURE 4. The drawings contrast the graphs of the estimation error made by the Supremum bound (in red) and the absolute estimation error made by the Sharp Estimate **S.E.** (in brown). Figures are drawn w.r.t. p_n .

Lemma 1.8 (The Supremum Bound on the Sum of Prime Numbers).

The Supremum bound on the sum of primes up to p_n , for all $p_n \in \mathbb{N} \mid p_n \geq 5$ is given by:

$$(1.8) \quad \text{SUP}_{p_n} = \int_2^{\theta_s} \frac{dt}{\log t} > \sum_{i=1}^n p_{(i)} = \mathcal{T.S.}$$

Where $\mathcal{T.S.}$ and θ_s are given by the Definitions: 1.1 and 1.2 respectively.

Proof.

Suppose that for $p_n \in \mathbb{N} \mid p_n \geq 17$ the inequality is false:

$$(1.9) \quad \left[\left(\left(\int_2^{\theta_s} \frac{dt}{\log t} \right) / \left(\sum_{i=1}^n p_{(i)} \right) \right) - 1 \right] > \frac{1}{p_n}$$

This implies that:

$$(1.10) \quad \left[\left(\left(\int_2^{\theta_s} \frac{dt}{\log t} \right) / \left(\sum_{i=1}^n p_{(i)} \right) \right) - 1 \right]^{(-1)} > p_n$$

However, at $p_n = 17$ the difference of terms of the inequality 1.10 attains approximately -1.3563587 and diverges at every step n , at the rate $\propto k p_n$ with $k \rightarrow 1$ as $p_n \rightarrow \infty$. Consequently, we have a contradiction. Necessarily therefore,

$$(1.11) \quad \left[\left(\left(\int_2^{\theta_s} \frac{dt}{\log t} \right) / \left(\sum_{i=1}^n p_{(i)} \right) \right) - 1 \right]^{(-1)} < p_n$$

for all $p_n \in \mathbb{N} \mid p_n \geq 17$ the inequality 1.11 holds. This in turn implies that for all $p_n \in \mathbb{N} \mid p_n \geq 17$,

$$(1.12) \quad \left(\int_2^{\theta_s} \frac{dt}{\log t} \right) > \sum_{i=1}^n p_{(i)} \left(1 + \frac{1}{p_n} \right) = \left(\sum_{i=1}^n p_{(i)} + \frac{\sum_{i=1}^n p_{(i)}}{p_n} \right)$$

Consequently, Lemma 1.8 holds for all $p_n \in \mathbb{N} \mid p_n \geq 17$. For all $5 \leq p_n \leq 17$ Table 1 lists all values of p_n within the range, demonstrating that Lemma 1.8 holds in this

range as well. Thus, necessarily Lemma 1.8 holds as stated for all $p_n \in \mathbb{N} \mid p_n \geq 5$, concluding the proof. $\square$

TABLE 1. Low Range Difference of $\mathcal{SUP}_{p_n}$ and $\mathcal{T.S.}$

p_n	$\mathcal{SUP}_{p_n} - \mathcal{T.S.}$	p_n	$\mathcal{SUP}_{p_n} - \mathcal{T.S.}$
3	-0.159632	11	2.25869
5	0.609876	13	2.81551
7	1.10942	17	3.70758

1.2. Sharp Estimate of the Sum of Primes.

Definition 1.9 (Sharp Estimate $\mathcal{S.E.}_{(p_n)}$ of the Sum of Prime Numbers).

$$(1.13) \quad \mathcal{S.E.}_{(p_n)} = \int_2^{\theta_{(e)}} \frac{dt}{\log t} \quad \forall p_n \in \mathbb{N} \mid p_{(n)} \geq 2$$

Lemma 1.10 (Bounds on Relative Error of the Sharp Estimate $\mathcal{S.E.}_{(p_n)}$).

The absolute relative error, of the Sharp Estimate of the sum of prime numbers up to p_n , for all $p_n \in \mathbb{N} \mid p_n \geq 2$ has the following bounds:

$$(1.14) \quad \mathcal{L.B.}_{(p_n)} = \frac{1}{(p_n)^2} < \mathcal{R.E.}_{(p_n)} = \frac{|\mathcal{E.E.}|}{\mathcal{T.S.}} < \frac{1}{\sqrt[3]{p_n}} = \mathcal{U.B.}_{(p_n)}$$

Where $\mathcal{S.E.}_{(p_n)}$, θ_e and $\mathcal{E.E.}$ are given by the Definitions 1.9, 1.3 and 1.4 respectively

While $\gamma \approx 0.57721566490153286061$ is the Euler-Mascheroni constant.

Proof.

Suppose that for $p_n \in \mathbb{N} \mid p_n \geq 2$, the following inequality is false:

$$(1.15) \quad \frac{\left| \left[\left(\int_2^{\theta_e} \frac{dt}{\log t} \right) - \mathcal{T.S.} \right] \right|}{\mathcal{T.S.}} - \frac{1}{(p_n)^2} > 0$$

This implies that it must be true:

$$(1.16) \quad \left(\frac{\left| \left[\left(\int_2^{\theta_e} \frac{dt}{\log t} \right) - \mathcal{T.S.} \right] \right|}{\mathcal{T.S.}} \right)^{(-1)} - (p_n)^2 > 0$$

However, at $p_n = 2$ the inequality 1.16 attains ~ -0.910865 and diverges rapidly at every step, at a rate $\propto p_n^2$. Consequently we have a contradiction to the initial hypothesis. Necessarily therefore, for all $p_n \in \mathbb{N} \mid p_n \geq 2$:

$$(1.17) \quad \frac{\left| \left[\left(\int_2^{\theta_e} \frac{dt}{\log t} \right) - \mathcal{T.S.} \right] \right|}{\mathcal{T.S.}} > \frac{1}{(p_n)^2}$$

the inequality 1.18 must be true. Now suppose, that for $p_n \in \mathbb{N} \mid p_n \geq 127$ the following inequality is false:

$$(1.18) \quad \frac{\left| \left[\left(\int_2^{\theta_e} \frac{dt}{\log t} \right) - \mathcal{T.S.} \right] \right|}{\mathcal{T.S.}} - \frac{1}{\sqrt[3]{p_n}} < 0$$

This implies that:

$$(1.19) \quad \left(\frac{\left| \left[\left(\int_2^{\theta_e} \frac{dt}{\log t} \right) - \mathcal{T.S.} \right] \right|}{\mathcal{T.S.}} \right)^{(-1)} - (\sqrt[3]{p_n}) < 0$$

However, at $p_n = 127$ inequality 1.19 attains ~ 45.218 , and diverges at every step, at a rate exceeding $\sqrt[3]{p_n}$. The divergence rate further increases as p_n increases unboundedly. Consequently we have a contradiction to the initial hypothesis. Thus necessarily, for all $p_n \in \mathbb{N} \mid p_n \geq 127$:

$$(1.20) \quad \frac{\left| \left[\left(\int_2^{\theta_e} \frac{dt}{\log t} \right) - \mathcal{T.S.} \right] \right|}{\mathcal{T.S.}} < \frac{1}{\sqrt[3]{p_n}}$$

Direct computer evaluation demonstrates that for p_n in the range $2 \leq p_n \leq 127$, inequality 1.20 holds. The output in graphical form is given in Figure 5. Thus Lemma 1.10 holds as stated. Concluding the proof. $\square$

Corollary 1.11 (The Relative Error $\mathcal{R.E.}_{(p_n)}$ of $\mathcal{S.E.}_{(p_n)}$ Limit).

Clearly, both the lower bound $\mathcal{L.B.}_{(p_n)}$ as well as the upper bound $\mathcal{U.B.}_{(p_n)}$ converge to zero as p_n increases unboundedly. Consequently this implies, that the relative error $\mathcal{R.E.}_{(p_n)}$ of $\mathcal{S.E.}_{(p_n)}$ given by Lemma 1.10, which is squeezed between those bounds, must converge to zero as well:

$$(1.21) \quad \lim_{p_n \rightarrow \infty} (\mathcal{LB}) = \lim_{p_n \rightarrow \infty} (|\mathcal{R.E.}_{(p_n)}|) = \lim_{p_n \rightarrow \infty} (\mathcal{UB}) \rightarrow 0$$

FIGURE 5. The linear drawing on the left shows the difference of terms of the $\mathcal{U.B.}_{(p_n)}$ and $\mathcal{R.E.}_{(p_n)}$, the figure is drawn w.r.t. $p_n \in \mathbb{N}$. The log-log drawing on the right shows the Relative Estimation Error $\mathcal{R.E.}_{(p_n)}$ (in blue), the lower bound $\mathcal{L.B.}_{(p_n)}$ (in green) and the upper bound $\mathcal{U.B.}_{(p_n)}$ (in red) over the range $p_n \in \mathbb{N} \mid 2 \leq p_n \leq 23879519$.

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2. APPENDIX

FIGURE 6. The drawing shows the graphs of the True Sum of primes (in black) and the Sharp Estimate **S.E.** (in red) w.r.t. p_n .