Nested Krylov methods for shifted linear systems

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**Motivation**

We consider the time-harmonic elastic wave equation at multiple frequencies $\omega_k$,

$$-\omega_k^2 \rho(x) u - \nabla \cdot \tau(u) = s, \quad x \in \Omega \subset \mathbb{R}^{2,3}. \tag{1}$$

Discretization of (1) including Sommerfeld boundary conditions leads to

$$(K + i\omega_k C - \omega_k^2 M) u = s,$$

which can be re-arranged to

$$\begin{bmatrix} iM^{-1}C & M^{-1}K \ 0 & I \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} \omega_k u \\ u \end{bmatrix} = \begin{bmatrix} M^{-1}I \\ 0 \end{bmatrix} s. \tag{2}$$

**Multi-shift Krylov methods**

Shifted linear systems like (2) are of the form

$$(A - \omega I)x_k = b, \quad k = 1, \ldots, N_\omega. \tag{3}$$

Idea for simultaneous solution: Krylov subspaces are shift-invariant, i.e.

$$K_m(A,b) \equiv \text{span}\{b, Ab, \ldots, A^{m-1}b\} = K_m(A - \omega I, b), \quad \text{for all } \omega \in \mathbb{C}. \tag{4}$$

**Nested preconditioners for shifted problems**

Preserve shift-invariance when preconditioning (3):

$$K_m(A P^{-1}, b) = K_m((A - \omega I) P_\omega^{-1}, b). \tag{5}$$

In [2], polynomial preconditioners of degree $n$ are discussed,

$$P_\omega^{-1} \equiv \sum_{i=1}^n \alpha_i A^i \approx (A - \omega I)^{-1}, \quad P_\omega^{-1} \equiv \sum_{i=1}^n \alpha_i(\omega) A^i \approx (A - \omega I)^{-1},$$

that are constructed such that shift-invariance in (4) is preserved.

**Nested Krylov preconditioners [1]**: Combine a flexible multi-shift Krylov method with a variable polynomial preconditioner.

- Use a Krylov polynomial as an inner preconditioner, e.g. ms_FOM.
- Truncation of inner method at $t \alpha_1 \sim 0.1$, see Fig. 3.
- Use preconditioner in a flexible outer Krylov iteration, e.g. ms_FGMRES.

**Results**

![Fig. 1: Full-waveform inversion.](image1)

![Fig. 2: Solution of (1) for a two-layered test case in time-domain (left) and frequency-domain (right).](image2)

![Fig. 3: Convergence curves of nested FOM-FGMRES for $N_\omega = 6$ frequencies.](image3)

**References**
