Introduction

We present the approximation of an optimal control problem for linear parabolic PDEs. The method is based on a model reduction technique using Proper Orthogonal Decomposition (POD-MOR). POD-MOR is a Galerkin approach where the basis functions are obtained upon information contained in time snapshots of the parabolic PDE related to given input data. We show that it is important to have knowledge about the controlled system at the right time instances. For the determination of the time instances (snapshot locations) we propose an a-posteriori error control which is based on a reformulation of the optimality system as a second order in time and fourth order in space elliptic system which is approximated by a space-time finite element method. Finally, we present numerical tests to illustrate our approach and to show the effectiveness of the method in comparison to existing approaches.

Problem Formulation

We deal with a standard linear quadratic problem where the cost functional we want to minimize is

$$ J(y, u) = \frac{1}{2} \| y - y_d \|_{L^2(0,T)}^2 + \frac{\alpha}{2} \| u \|_{L^2(Q)}^2 $$

subject to the following constraint

$$ \begin{cases} y_t(x, t) - \Delta y(x, t) = u(x, t) + f(x, t) & \text{in } \Omega \times (0,T) \\ y(x, 0) = y_0(x) & \text{on } \partial \Omega \times (0,T) \\ y(x, t) = 0 & \text{on } \partial \Omega \times (0, T) \end{cases} $$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary, $y_0 \in H^1(\Omega)$, $\alpha, T > 0$, $W = L^2(\Omega)$ and $u \in L^2(Q)$. Optimality system:

1. $y_t = 0 + f$ in $\Omega_t$, $y = 0$ on $\Sigma_T$, $y(0) = y_0$, $\alpha, T > 0$
2. $-y_{tt} - \Delta y = 0$ in $\Omega_T$, $p = 0$ on $\Sigma_T$, $p(T) = 0$
3. $\alpha u + p = 0$ in $\Omega_T$

Numerical Test

Choose data: $\Omega = (0, 1)$, $T = 1$, $\alpha = 1$, $y_0(x, t) = \sin(\pi x)\tan(\theta(t-1/2)/\varepsilon)$, $u(x, t) = \sin(x)$.$(\pi \cos(\pi t) - \pi^2 \sin(\pi t) + \tan(\theta(t-1/2)/\varepsilon))$

Figure 1: Adaptive space-time grids with dof=21 and $\Delta x = 1/7$ (left) and $\Delta x = 1/100$ (right) calculated by utilizing the temporal residual type a-posteriori error estimation for $y$. Obviously, temporal and spatial discretization decoupled which motivates to use a large spatial discretization size in the offline computation phase in Algorithm 1 in order to achieve a fast online stage.

Figure 2: True optimal state $u_{opt}$ (left), approximated POD solution computed on an equivalent time grid (middle) and approximated POD solution utilizing the adaptive time grid (right) with dof=21.

Figure 3: Contour lines of the true optimal state $u_{opt}$ (left), approximated POD solution computed on an equivalent time grid (middle) and approximated POD solution utilizing the adaptive time grid (right) with dof=21.

Table 1: Absolute errors between the true optimal solution and the POD suboptimal solution depending on the time discretization (equidistant: columns 1-3, adaptive: columns 4-6).

Numerical Strategy

Algorithm 1 Adaptive snapshot selection for optimal control problems.

Require: (Large) space step size $\Delta x$, number degree of freedom (DOF) for the time discretization, $T > 0$.
1. Compute a numerical approximation of (4) with temporal a-posteriori error estimation.
2. Obtain a time grid and an approximation of the optimal control.
3. Use this control to build the snapshot matrix $V$.
4. Compute a POD basis of order $\ell$.
5. Set up and solve the reduced problem on the time adaptive grid.

References