Reduced Basis Approximations for Maxwell's Equations in Dispersive Media

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Motivation

Simulation of **electromagnetic** and **optical** wave propagation in, e.g. water, fog or dielectric waveguides requires modeling of linear, temporally dispersive media. Using a POD-greedy sampling driven by an error indicator, we seek to generate a reduced model which accurately captures the dynamics. Typically, the reduced basis model reduction reduces the model order by a factor of more than 100, while maintaining an approximation error of less than 1%, [1,2].

Simulation

The time-harmonic Maxwell's equations in the electric field E are coupled with the polarization P

$$\frac{1}{\mu_0} \nabla \times \nabla \times E + \epsilon_0 \epsilon_\infty \partial_t^2 E = f - \partial_t^2 P,$$

$$\partial_t P + \frac{1}{\tau} P = \frac{\epsilon_0(\epsilon_s - \epsilon_\infty)}{\tau} E,$$

which is derived from adding relaxation polarization to the instantaneous polarization $\epsilon_0 \epsilon_r E(t, \mathbf{x})$, namely

$$\epsilon_0 \epsilon_\infty E(t, \mathbf{x}) + \epsilon_0 \int_{-\infty}^t E(t - \tau, \mathbf{x}) \chi(\tau) d\tau.$$

This is called a Debye relaxation and models polar molecules with a permanent dipole moment, [3]. Under the application of an external field, the molecules rotate and cause friction, which leads to an exponential damping.

Material parameters:

- relative permittivity in the static frequency limit ϵ_s
- relative permittivity in the infinite frequency limit ϵ_∞
- relaxation time τ
- vacuum permittivity ϵ_0 and vacuum permeability μ_0

Typical material values are for instance for water $\epsilon_\infty = 1.80$, $\epsilon_s = 81.00$, $\tau = 9.400 \times 10^{-12}$ s .

Simulation details:

- unit square with zero Dirichlet boundary conditions (PEC - perfectly electric conducting)
- broadband input source f modeled as a Gaussian pulse
- computational mesh has 9'680 degrees of freedom, Nédélec edge elements
- Runge-Kutta-Nyström time-stepping in the electric field
- explicit Euler in the polarization

Wave propagation of one component of the electric field after normalization.

References

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Parametric Model Order Reduction

The model is parametrized in τ and $\Delta_\epsilon = \epsilon_s - \epsilon_\infty$, defining the 2-dimensional parameter domain D . We seek to generate a reduced model which accurately captures the dynamics in the parameter domain D over timescales of a few nanoseconds.. A computationally intensive offline phase determines the projection space V_N , the parameter-independent parts of the affine form and the quantities required for error estimation. The online phase solves the reduced order PDE and evaluates the error estimator independently of the originally large system dimension.

It holds

$$\Delta(\nu) = \|E(\cdot, \nu) - E_N(\cdot, \nu)\| \leq \sum_{k=1}^n a^k b_{n-k},$$

where

$$a = \sup_{n \in \{1, \dots, n_T\}} \sup_{w \neq y} \frac{\|w - y\|}{\|f(w, t_n; \nu) - f(y, t_n; \nu)\|},$$

is the inverse Lipschitz constant and b_n is the residual at timestep n . Rigorous error estimation requires an upper bound on the inverse Lipschitz constant, since this is extremely costly, we use $a = 1$ here.

POD-Greedy Algorithm

- sampled parameter domain D in τ and $\Delta_\epsilon = \epsilon_\infty - \epsilon_s$
- solve model at parameter configuration ν_1 for $E(t_i; \nu_1)$ and $P(t_i; \nu_1)$, where $i = 1, \dots, n_T$ and set $k = 1$
- POD of trajectory $\Rightarrow$ projection basis V_N
- while $\max_{\nu} \Delta(\nu) > \text{tolerance}$
- set $k = k + 1$;
- set $\nu_k = \arg \max_{\nu} \Delta(\nu)$
- solve model for trajectory $E(\cdot; \nu_k)$ and $P(\cdot; \nu_k)$
- POD of trajectory $\Rightarrow$ append modes to V_N
- end while

Interpolatory Decomposition

Since the singular value decomposition (SVD) in the POD step is costly, an interpolatory decomposition (ID) is considered as an alternative, [3]. The potential advantage is, that computation times are lower than for an SVD. On the other hand, the interpolatory decomposition does not generate orthonormal matrices. This model also includes the adaptive snapshot selection [5] to enhance the offline times by compressing the trajectories before the SVD.

The interpolatory decomposition of a matrix $A \in \mathbb{R}^{m \times n}$ is a decomposition into $U \in \mathbb{R}^{m \times k}$, $B \in \mathbb{R}^{k \times k}$ and $V \in \mathbb{R}^{n \times k}$ as

$$A \approx U \circ B \circ V^T,$$

where B is a $k \times k$ submatrix of A . It expresses each of the columns of A as a linear combination of k selected columns of A and analogously for the rows. This selection defines the $k \times k$ submatrix B of A , and in the resulting system of coordinates, the action of A is represented by the action of its submatrix B .

Numerical Results

Method	500 timesteps	1500 timesteps	3000 timesteps
SVD	2.9 s	10.9 s	26.6 s
ID	0.9 s	4.4 s	13.2 s

Comparison of SVD and ID computation times for different number of timesteps.

Comparison of SVD (red) and ID (blue) generated models. Shown is the maximum relative error in the electric field over the sampled parameter domain.