A Hierarchical A-Posteriori Error Estimator for the Reduced Basis Method

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Introduction

We present an online-efficient hierarchical a-posteriori error estimator for the Reduced Basis Method (RBM), which uses the difference of two reduced approximations of different accuracy to estimate the true error of a reduced approximation. Effectivity as well as performance are investigated, especially for those cases where the inf-sup constant has small values and/or is hard to compute numerically.

Reduced Basis Method (RBM)

Assumptions

- X, Y Hilbert spaces and $\mathcal{P} \subset \mathbb{R}^P$ parameter space,
- $a : \mathcal{P} \rightarrow [X^{\mathcal{N}} \times Y^{\mathcal{N}} \rightarrow \mathbb{C}]$ uniformly bounded and uniformly inf-sup stable sesquilinear form with inf-sup constant $\beta^{\mathcal{N}}(\mu)$, where $\mu \in \mathcal{P}$.
- $\beta_{\text{LB}}^{\mathcal{N}}(\mu)$ computable lower bound of $\beta^{\mathcal{N}}(\mu)$.
- $f : \mathcal{P} \rightarrow [Y^{\mathcal{N}} \rightarrow \mathbb{C}]$ uniformly bounded linear form.

High-fidelity Problem

$X^{\mathcal{N}} \subset X$, $Y^{\mathcal{N}} \subset Y$ with $\dim(X^{\mathcal{N}}) = \dim(Y^{\mathcal{N}}) = \mathcal{N} < \infty$.

- For $\mu \in \mathcal{P}$ find $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$, s.t.

$$a(u^{\mathcal{N}}(\mu), v^{\mathcal{N}}; \mu) = f(v^{\mathcal{N}}; \mu) \quad \forall v^{\mathcal{N}} \in Y^{\mathcal{N}}.$$

Reduced Problem

$X_N \subset X^{\mathcal{N}}$ RB space and $Y_N \subset Y^{\mathcal{N}}$ stabilized RB space with $\dim(X_N) = \dim(Y_N) = N \ll \mathcal{N}$.

- For $\mu \in \mathcal{P}$ find $u_N(\mu) \in X_N$, s.t.

$$a(u_N(\mu), v_N; \mu) = f(v_N; \mu) \quad \forall v_N \in Y_N. \quad (1)$$

Residual Based Error Estimator (see e.g. [7, 8])

$$\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X \leq \frac{\|R_N^{\mathcal{N}}(\cdot; \mu)\|_{Y'}}{\beta_{\text{LB}}^{\mathcal{N}}(\mu)} =: \Delta_N^{\text{Res}}(\mu),$$

where $R_N^{\mathcal{N}}(v^{\mathcal{N}}; \mu) := f(v^{\mathcal{N}}; \mu) - a(u_N(\mu), v^{\mathcal{N}}; \mu)$.

Saturation Property

- X_N and X_M RB Spaces with $X_N \subsetneq X_M \subset X^{\mathcal{N}}$.

Saturation property: If $\Theta_{N,M}^{\mathcal{N}} \in [0, 1)$ exists, s.t.

$$\|u^{\mathcal{N}}(\mu) - u_M(\mu)\|_X \leq \Theta_{N,M}^{\mathcal{N}} \|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X \quad \forall \mu \in \mathcal{P}.$$

Outlook/Remarks

- Offline approximation of $\Theta_{N,M}^{\mathcal{N}}$.
- $\Delta_{N,M}(\mu)$ was applied to first order transport equations, see [3].

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A Hierarchical A-Posteriori Error Estimator

Preliminaries

- For $X_N \subsetneq X_M \subset X^{\mathcal{N}}$ let $u_N(\mu) \in X_N$, $u_M(\mu) \in X_M$ solutions of (1) and define the hierarchical term

$$\Delta_{N,M}(\mu) := \|u_N(\mu) - u_M(\mu)\|_X.$$

Asymptotic Approach

- The triangle inequality yields

$$\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X \leq \|u^{\mathcal{N}}(\mu) - u_M(\mu)\|_X + \|u_M(\mu) - u_N(\mu)\|_X \leq \Delta_M^{\text{Res}}(\mu) + \Delta_{N,M}(\mu).$$

- If *Kolmogorov M-width* decays, X_M may be constructed s.t. $\Delta_M^{\text{Res}}(\mu) \leq \varepsilon \Delta_{N,M}(\mu)$ for all μ , see [2]. Therefore

$$\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X \leq (1 + \varepsilon) \Delta_{N,M}(\mu).$$

Approach by using the Saturation Property

- If the *saturation property* holds, it is well-known, see e.g. [1, 5], that

$$\frac{\Delta_{N,M}(\mu)}{1 + \Theta_{N,M}^{\mathcal{N}}} \leq \|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X \leq \frac{\Delta_{N,M}(\mu)}{1 - \Theta_{N,M}^{\mathcal{N}}} =: \Delta_{N,M}^{\text{Hier}}(\mu).$$

- For the offline approximation of $\Theta_{N,M}^{\mathcal{N}}$ we suggest to decompose $\mathcal{P}$ in compact subsets $\mathcal{P}_i$ such that on each subset $\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X \neq 0$ be valid. Solving the nonlinear problem, see [4],

$$\Theta_{N,M,i}^{\mathcal{N}} := \arg \min_{q \in \mathbb{R}_{\geq 0}} |F_i(q)| \quad \text{with} \quad F_i(q) := \max_{\mu \in \mathcal{P}_i} \left\{ \|u^{\mathcal{N}}(\mu) - u_M(\mu)\|_X - q \cdot \|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X \right\} \quad \text{!}$$

with a suitable initial value, we define $\Theta_{N,M}^{\mathcal{N}} := \max_i \Theta_{N,M,i}^{\mathcal{N}}$.

- For the RB space X_M we suggest a Taylor-RB method, see e.g. [8], i.e.

$$X_M := \text{span} \left\{ u^{\mathcal{N}}(\mu_n), \frac{\partial^k}{\partial \mu_i^k} u^{\mathcal{N}}(\mu_n) : k = 1, \dots, K_n, i = 1, \dots, P, n = 1, \dots, N \right\}, \quad K_n \in \mathbb{N}_0.$$

- For the effectivity factor $\eta_{N,M}^{\mathcal{N}}(\mu) := \Delta_{N,M}^{\text{Hier}}(\mu) / \|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X$ we get

$$1 \leq \eta_{N,M}^{\mathcal{N}}(\mu) \leq \frac{1 + \Theta_{N,M}^{\mathcal{N}}}{1 - \Theta_{N,M}^{\mathcal{N}}}.$$

Numerical Examples

Model Problem

- Let $\mathcal{P}^{(1)} := [95, 100]$ and $\mathcal{P}^{(2)} := [90, 100]$.
- For $\mu \in \mathcal{P}^{(1)}$ or $\mu \in \mathcal{P}^{(2)}$ consider Helmholtz-Problem (1D)

$$\begin{aligned} -u''(x) - \mu^2 u(x) &= 1, & x \in (0, 1), \\ u(0) &= 0, \\ u'(1) + i\mu u(1) &= 0. \end{aligned}$$

Figure 1: SCM-convergence on $\mathcal{P}^{(1)}$.

Numerical Results - Error Plot for Model Problem

- Parameter sampling with $\Delta_{N,M}(\mu)$.
- Computation of $\beta^{\mathcal{N}}(\mu)$ with generalized eigenvalue problem.
- Computation of $\Theta_{N,M}^{\mathcal{N}}$ using $\sup_{\mu \in \mathcal{P}_{\text{train}}} \frac{\|u^{\mathcal{N}}(\mu) - u_M(\mu)\|_X}{\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X}$. !
- $\frac{\sum_{i=1}^{100} \|u^{\mathcal{N}}(\mu_i) - u_N(\mu_i)\|_X}{100}$, $\frac{\sum_{i=1}^{100} \Delta_{N,M}^{\text{Std}}(\mu_i)}{100}$ and $\frac{\sum_{i=1}^{100} \Delta_{N,M}^{\text{Hier}}(\mu_i)}{100}$ are shown in Figure 2 and 3, where $\mu_1, \dots, \mu_{100}$ are randomly chosen.

Figure 2: Parameter space $\mathcal{P}^{(1)}$ using $K_n = 2$ for all $n \in \mathbb{N}$.

Figure 3: Parameter space $\mathcal{P}^{(2)}$ using $K_n = 3$ for all $n \in \mathbb{N}$.

Numerical Results - Online CPU-Time for Model Problem

- $\beta_{\text{LB}}^{\mathcal{N}}(\mu)$ computed with Successive Constraint Method (SCM), e.g. [6].
- SCM did not converge on $\mathcal{P}^{(2)}$. Therefore we consider only $\mathcal{P}^{(1)}$.
- Lagrange-type basis construction with strong greedy for X_N and X_M .
- $\eta^{\mathcal{N}}(\mu)$ controlled by SCM-tolerance, $\dim(X_N)$ and/or $\dim(X_M)$. Associated online CPU-time measured for randomly chosen parameter.