Approximation in an $M/G/1$ queueing system with breakdowns and repairs

Karim ABBAS and Djamil AÏSSANI
Laboratory of Modelization and Optimization of Systems
University of Bejaia, 06000(Algeria)
karabbas2003@yahoo.fr

Abstract

In this paper, we study the strong stability in the $M/G/1$ queueing system with breakdowns and repairs after perturbation of the breakdown's parameter. Using the approximation conditions in the classical $M/G/1$ system, we obtain stability inequalities with exact computation of the constants. Thus, we can approximate the characteristics of the $M/G/1$ queueing system with breakdowns and repairs by the classical $M/G/1$ corresponding ones.

Keywords: $M/G/1$ queueing system, Breakdowns, Perturbation, Approximation, Strong stability, Stability inequalities.

1. INTRODUCTION

In classical queueing models, it is common to assume that the service station have not failed. However, in practice we often meet cases where the service station may experience breakdowns. Therefore, a more realistic queueing model is a model which its service station is unable from time to time to deliver service to its customers. Such models are useful to investigate the performance of unreliable computer systems, manufacturing systems, etc. The impact of these service breakdowns on the performance of a given queueing system will depend on the specific interaction between the breakdowns process and the service process. Therefore, such systems cannot, in general, be analyzed as the reliable one.

Unreliable server queues have been developed tremendously since the pioneering work of Thiruvengadam [1] and Avi-Itzhak and Naor [2]. Li and al. [3] and Tang [4] investigated the behavior of the unreliable server, and the effect of server breakdowns and repairs in the $M/G/1$ queueing models from both the queueing and the reliability points of view. Wang and al. [5] investigated the controllable $M/H_k/1$ queueing system with unreliable server. They developed analytic closed-form solutions and provided a sensitivity analysis. Pearn and al. [6] have obtained analytical results for sensitivity analysis of the optimal management policy for $M/G/1$ queueing system with removable and non-reliable server. Wang [7] has used a supplementary variable method for analysis of an $M/G/1$ queue with second optional service and server breakdowns. Wang and al. [8] have used maximum entropy principle to derive the approximate formulas in a single removable and unreliable server in the N policy $M/G/1$ queueing system with general startup times. Lam and al. [9] has used a supplementary variable method for analysis of a geometric process model for $M/M/1$ queueing system with repairable service station. Recently, Ke and Lin [10] developed an approach that can provide fuzzy system characteristics for queues which are described and solved by parametric nonlinear programming.

Therefore, analytic closed-form solutions could be obtained only for certain special unreliable models. Nevertheless, even in these cases, the complexity of these analytical formulas does not allow exploiting them in the practice, that is the case of the generating function of the various limiting distributions or Laplace transform (see for example [11]). For this reason, there exists, when modelling a real system, a common technique for substituting the real but complicated elements governing a queueing system by simpler ones in some sense close to the real elements.

The queueing model so constructed represents an idealization of the real queueing one, and hence the "stability" problem arises.

A queueing system is said to be stable if small perturbations in its parameters entail small perturbations in its characteristics [12, 13]. So, the margin between the corresponding characteristics of two stable queueing systems is obtained as function of the margin between their parameters.

The strong stability method elaborated in the beginning of the 1980s [13, 14] permits us to investigate the ergodicity and the stability of the stationary and non-stationary characteristics of Markov chains [12]. In contrast to other methods, we assume that the perturbations of the transition kernel are small with respect to some norms in the operator space. Such a strict condition allows us to obtain better stability estimates and enables us to find precise asymptotic expressions of the characteristics of the "perturbed" system [15].

The remainder of this paper is composed as follows: In section 2, we present the considered queueing models and we pose the problem. Notations and preliminaries are introduced in Section 3. The main results of this paper are given in Section 4. After perturbing the breakdown's parameter, we estimate the deviation of its transition operator, and we give an upper bound to the approximation error (Section 4).

2. MODELS DESCRIPTION AND ASSUMPTIONS

We consider an $M/G/1 (FIFO, \infty)$ queueing system with a repairable service station. Customers arrive according to a Poisson process with rate λ and demand an independent and identically distributed service times with common distribution function B with mean $1/\mu$. We shall assume that when a server fails, the time required to repair it have exponential distribution with rate $r > 0$.

In this model, we consider the breakdowns with losses. If an arriving customer finds the server idle-up (i.e. it is ready to serve), it will be immediately occupy the server with probability $q > 0$ and leaves it after service completion if no breakdown had occurred during this period. If the server is failed, then this blocked customer may leave the system altogether without being served with probability $1 - q$. We assume that when the server fails, it is under repair and it cannot be occupied. As soon as the repair of the failed server is completed, the server enters an operating state and continues to serve the other customers immediately.

The state of $M/G/1$ queueing system with a repairable service station at time t , can be described by the following stochastic process,

$$S(t) = \{X(t), e(t), K(t); t \geq 0\}, \quad (1)$$

where $X(t) \in \mathbb{N}$ is the number of customers in the system at time t and $\{e(t), K(t)\}$ are two supplementary state variables. $e(t) \in \{0, 1\}$ describes global properties (up or failed) of the server:

$$e(t) = \begin{cases} 0, & \text{if the server is available for service at time } t, \\ 1, & \text{if the server is failed at time } t. \end{cases} \quad (2)$$

$K(t)$ is a random variable, which is defined below:

- If $e(t) = 0$ and $X(t) = 0$, then $K(t)$ is equal to the time elapsed since the time t to the occurrence time of a breakdown, if after t the arrivals are interrupted.
- If $X(t) \neq 0$,
 - If $e(t) = 0$, $K(t)$ is the duration of the remaining service period.
 - If $e(t) = 1$, $K(t)$ is the duration of the remaining repair period.

Clearly, $S(t)$ is a Markov process. The computation of its transitional regime intervenes the integro-differential equations. For that, we use the method of the embedded Markov chain.

We use the following notations:

- γ_n : are the times of "end" of the n^{th} service, or "end" of the n^{th} repair.
- Δ_n : is a random variable, that represents the number of the arrivals during the n^{th} service time, or during the n^{th} repair time.
- $X_n = X(\gamma_n)$: is number of customers in the system immediately after the "end" of the n^{th} service time, or immediately after the "end" of the n^{th} repair time.

$$\delta_{X_n} = \begin{cases} 0, & \text{if } X_n = 0; \\ 1, & \text{if } X_n > 0. \end{cases} \quad (3)$$

$$\mathbb{I}_{\{e_n=k\}} = \begin{cases} 1, & \text{if } e_n = k; \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Where $k \in \{0, 1\}$.

Lemma 2.1. *The sequence $S_n = S(\gamma_n)$ forms a Markov chain, its transition operator $P = (P_{ij})_{i,j \geq 0}$ is defined by,*

$$P_{ij} = \begin{cases} q \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x) + \frac{r(1-q)}{\lambda+r} \left(\frac{\lambda}{\lambda+r}\right)^j, & \text{if } i = 0; \\ q \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} dB(x) + \frac{r(1-q)}{\lambda+r} \left(\frac{\lambda}{\lambda+r}\right)^{j-i+1}, & \text{if } 0 < i \leq j+1; \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Proof. To calculate the probabilities $(P_{ij})_{i,j \geq 0}$, we consider all the following possible events:

- (1) $\{X_{n+1} = j \text{ and } e_{n+1} = 0\}$, given that, $\{X_n = i \text{ and } e_n = 0\}$; $i, j \geq 0$.
- (2) $\{X_{n+1} = j \text{ and } e_{n+1} = 1\}$, given that, $\{X_n = i \text{ and } e_n = 0\}$; $i, j \geq 0$.
- (3) $\{X_{n+1} = j \text{ and } e_{n+1} = 0\}$, given that, $\{X_n = i \text{ and } e_n = 1\}$; $i, j \geq 0$.

We have,

$$X_{n+1} = X_n + \Delta_{n+1} \mathbb{I}_{\{e_{n+1}=k\}} - \delta_{X_n}; k = \overline{0, 1}. \quad (6)$$

Therefore the transition probabilities P_{ij} are written under the following form:

$$\begin{aligned} P_{ij} &= \mathbb{P}r[X_n + \Delta_{n+1} \mathbb{I}_{\{e_{n+1}=0\}} - \delta_{X_n} = j / X_n = i] \\ &\quad + \mathbb{P}r[X_n + \Delta_{n+1} \mathbb{I}_{\{e_{n+1}=1\}} - \delta_{X_n} = j / X_n = i] \\ &= q \mathbb{P}r[\Delta_{n+1} \mathbb{I}_{\{e_{n+1}=0\}} = j - i + \delta_{X_n}] \\ &\quad + (1 - q) \mathbb{P}r[\Delta_{n+1} \mathbb{I}_{\{e_{n+1}=1\}} = j - i + \delta_{X_n}]. \end{aligned}$$

- If $X_n = i \neq 0$, then,

$$P_{ij} = q \mathbb{P}r[\Delta_{n+1} \mathbb{I}_{\{e_{n+1}=0\}} = j - i + 1] + (1 - q) \mathbb{P}r[\Delta_{n+1} \mathbb{I}_{\{e_{n+1}=1\}} = j - i + 1]. \quad (7)$$

- If $X_n = i = 0$, then,

$$P_{0j} = q \mathbb{P}r[\Delta_{n+1} \mathbb{I}_{\{e_{n+1}=0\}} = j] + (1 - q) \mathbb{P}r[\Delta_{n+1} \mathbb{I}_{\{e_{n+1}=1\}} = j]. \quad (8)$$

Where, the distribution of Δ_n is given by

- If $e_n = 0$, then,

$$\mathbb{P}r[\Delta_n \mathbb{I}_{\{e_n=0\}} = k] = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} \mathbb{I}_{\{e_n=0\}} dB(x). \quad (9)$$

- If $e_n = 1$, then,

$$\begin{aligned}\mathbb{P}r[\Delta_n \mathbb{I}_{\{e_n=1\}} = k] &= r \int_0^\infty e^{-(\lambda+r)x} \frac{(\lambda x)^k}{k!} \mathbb{I}_{\{e_n=1\}} dx \\ &= \frac{r}{\lambda+r} \left(\frac{\lambda}{\lambda+r} \right)^k.\end{aligned}$$

To calculate the operator transition, we distinguish the following cases:

(1) If the system is empty (i.e., $X_n = i = 0$), then,

$$P_{0j} = q \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} \mathbb{I}_{\{e_{n+1}=0\}} dB(x) + \frac{r(1-q)}{\lambda+r} \left(\frac{\lambda}{\lambda+r} \right)^j. \quad (10)$$

(2) If there are no arrivals during the service duration of $(n+1)^{th}$ customer, or during the repair duration of the server just after the departure of the n^{th} customer, then,

$$P_{ij} = q \int_0^\infty e^{-\lambda x} \mathbb{I}_{\{e_{n+1}=0\}} dB(x) + \frac{r(1-q)}{\lambda+r}.$$

(3) If $j > i - 1$, then,

$$P_{ij} = q \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} \mathbb{I}_{\{e_{n+1}=0\}} dB(x) + \frac{r(1-q)}{\lambda+r} \left(\frac{\lambda}{\lambda+r} \right)^{j-i+1}.$$

(4) If $j < i - 1$, then,

$$P_{ij} = 0.$$

Consider also an $M/G/1$ ($FIFO, \infty$) system with null breakdown's rate. It behaves exactly as a classical $M/G/1$ system: Arrivals occur as a Poisson process of rate λ , and it has the same general service time distribution function B . Let $\bar{P}$ be the transition operator for the corresponding Markov chain $\bar{X}_n$, in the classical $M/G/1$ system. We have [16]

$$\bar{P}_{ij} = \begin{cases} \int_0^\infty \frac{(\lambda x)^j}{j!} e^{-\lambda x} dB(x), & \text{if } j \geq 0, i = 0; \\ \int_0^\infty \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} e^{-\lambda x} dB(x), & \text{if } 1 \leq i \leq j+1; \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

We propose to obtain the stability inequalities, and this by applying the strong stability criterion [13] for the $M/G/1$ queueing system with null breakdown's rate. Indeed, the approximation conditions in the classical $M/G/1$ queueing system have been made [17] at the time of perturbation of the retrial's parameter in the $M/G/1/1$ retrial queue. By making tend the retrial's rate to ∞ and the resultant system is exactly the $M/G/1$ system with infinite retrials (or the classical $M/G/1$ system). Also, in our case, we make tend the breakdown's rate to zero (or we make tend the probability q to one) in the $M/G/1$ with breakdowns and repairs, so we will have the classical $M/G/1$ system. Therefore, we can consider the two perturbed systems, the $M/G/1$ with repairable station and the retrial $M/G/1$ queue, as two sequences which have the same limit.

3. THE STRONG STABILITY CRITERIA

3.1. PRELIMINARIES AND NOTATIONS

Let $\mathcal{M} = \{\mu_j\}$ be the space of finite measures on $\mathbb{N}$, and $\eta = \{f(j)\}$ the space of bounded measurable functions on $\mathbb{N}$. We associate with each transition kernel P the linear mapping:

$$(\mu P)_k(j) = \sum_{i \geq 0} \mu_i P_{ik}(j). \quad (12)$$

$$Pf(k) = \sum_{i \geq 0} f(i)P_{ki}. \quad (13)$$

Introduce on $\mathcal{M}$ the class of norms of the form:

$$\|\mu\|_v = \sum_{j \geq 0} v(j)|\mu_j|. \quad (14)$$

Where v is an arbitrary measurable function (not necessary finite) bounded below away from a positive constant. This norm induces in the space η the norm:

$$\|f\|_v = \sup_{k \geq 0} \frac{|f(k)|}{v(k)}. \quad (15)$$

Let us $\mathcal{B}$, the space of linear operators, with norm:

$$\|P\|_v = \sup_{k \geq 0} \frac{1}{v(k)} \sum_{j \geq 0} v(j)|P_{kj}|. \quad (16)$$

Definition 3.1. The Markov chain X with a transition kernel P and an invariant measure π is said to be strongly v -stable with respect to the norm $\|\cdot\|_v$, if $\|P\|_v < \infty$ and each stochastic kernel Q on the space $(\mathbb{N}, \mathcal{B}(\mathbb{N}))$ in some neighborhood $\{Q : \|Q - P\|_v < \varepsilon\}$ has a unique invariant measure $\nu = \nu(Q)$ and

$$\|\nu - \pi\|_v \rightarrow 0 \text{ as } \|Q - P\|_v \rightarrow 0. \quad (17)$$

Theorem 3.1. ([14]) The Markov chain X with the transition kernel P and invariant measure π is strongly v -stable with respect to the norm $\|\cdot\|_v$ if and only if there exist a measure α and a nonnegative measurable function h on $\mathbb{N}$ such that $\pi h > 0$, $\alpha \mathbf{1} = 1$, $\alpha h > 0$, and

- a) The operator $T = P - h \circ \alpha$ is nonnegative.
- b) There exists $\rho < 1$ such that $Tv(k) \leq \rho v(k)$ for $k \in \mathbb{N}$.
- c) $\|P\|_v < \infty$.

Here $\mathbf{1}$ is the function identically equal to 1.

Theorem 3.2. ([18]) Let X be a strongly v -stable Markov chain that satisfies the conditions of theorem (3.1). If ν is the measure invariant of a kernel Q , then for the norm $\|Q - P\|_v$ sufficiently small, we have

$$\nu = \pi[I - \Delta R_0(I - \Pi)]^{-1} = \pi + \sum_{t=1}^{\infty} \pi[\Delta R_0(I - \Pi)]^t. \quad (18)$$

Where $\Delta = Q - P$, $R_0 = (I - T)^{-1}$ and $\Pi = \mathbf{1} \circ \pi$.

corollary 3.1. Under the conditions of Theorem (3.1), we have

$$\nu = \pi + \pi \Delta R_0(I - \Pi) + o(\|\Delta\|_v^2), \quad (19)$$

as $\|\Delta\|_v \rightarrow 0$

corollary 3.2. Under the conditions of Theorem (3.1), for $\|\Delta\|_v < \frac{(1-\rho)}{c}$, we have the estimate

$$\|\nu - \pi\|_v \leq \|\Delta\|_v c \|\pi\|_v (1 - \rho - c\|\Delta\|_v)^{-1}. \quad (20)$$

Where

$$c = m\|P\|_v^{m-1}(1 + \|\mathbf{1}\|_v \|\pi\|_v);$$

$$\|\pi\|_v \leq (\alpha v)(1 - \rho)^{-1}(\pi h)m\|P\|_v^{m-1}.$$

3.2. STABILITY CONDITIONS IN THE CLASSICAL $M/G/1$ QUEUE

The first step consists of determination of the strong v -stable conditions of the $M/G/1$ system with null breakdown's rate. In fact, these conditions are already made by [17] at the time of perturbation of the retrial's parameter in the $M/G/1/1$ retrial queue. So, to apply Theorem (3.1) to the Markov chain $\bar{X}_n$, we choose the function $v(k) = \beta^k$, $\beta > 1$,

$$h_i = \delta_{i0} = \begin{cases} 1, & \text{if } i = 0; \\ 0, & \text{if } i > 0. \end{cases}$$

and $\alpha_j = \bar{P}_{0j}$.

We have the following result.

Theorem 3.3. [17] *Suppose that geometric ergodicity and the Cramér conditions in the system $M/G/1$ hold:*

- (1) $\lambda \mathbb{E}(\xi) < 1$, *where ξ is the service time,*
- (2) $\exists a > 0, \mathbb{E}(e^{a\xi}) = \int_0^\infty e^{au} dB(u) < \infty$ *and*

$$\beta_0 = \sup\{\beta : \hat{f}(\lambda\beta - \lambda) < \beta\}. \quad (21)$$

Where $\hat{f}(\lambda\beta - \lambda) = \int_0^\infty \exp[(\lambda\beta - \lambda)x] dB(x)$. Then, for all β such that $1 < \beta < \beta_0$, the Markov chain $\bar{X}_n$ is strongly stable for the test function $v(k) = \beta^k$.

The embedded Markov chain $\bar{X}_n$ being strongly stable then, the characteristics of the $M/G/1$ queueing system with breakdowns and repairs can be approximated by the corresponding ones of the classical $M/G/1$ queue.

4. STABILITY INEQUALITIES IN THE CLASSICAL $M/G/1$ QUEUE

In this section we obtain the inequalities of stability.

4.1. ESTIMATION OF THE TRANSITION KERNEL DEVIATION

To be able to estimate numerically the margin between the stationary distributions of the Markov chains $\bar{X}_n$ and X_n , we estimate the norm of the deviation of the transition kernel $\bar{P}$, and we use the following lemma:

Lemma 4.1. [19] *Let us suppose that $\int_0^\infty x|B - E|(dx) < \frac{W}{\lambda}$ where $W = W(B, E) = \int_0^\infty |B - E|(dx)$ and that it exists $a > 0$ such as $\int_0^\infty e^{ax}|B - E|(dx) < \infty$. Then, it exists $\beta > 1$ such as*

$$\int_0^\infty e^{\lambda(\beta-1)x}|B - E|(dx) < \beta W. \quad (22)$$

Theorem 4.1. *Let P (respectively $\bar{P}$) be the transition operator of the embedded Markov chain in the $M/G/1$ queueing system with breakdowns and repairs (respectively in the classical $M/G/1$ system). Let us suppose that the conditions of the Lemma (4.1) are verified. Then, for all β such that $1 < \beta < \beta_0$, we have*

$$\|P - \bar{P}\|_v \leq \beta_0 W, \quad (23)$$

where $W = W(B, E) = \int_0^\infty |B - E|(dx)$ and $\beta_0 = \sup\{\beta : \frac{\hat{f}(\lambda\beta - \lambda)}{\beta} < 1\}$.

Proof. From (16), we have

$$\|P - \bar{P}\|_v = \sup_{k \geq 0} \frac{1}{\beta^k} \sum_{j \geq 0} \beta^j |P_{kj} - \bar{P}_{kj}|, \quad (24)$$

(i) For $k = 0$:

$$\begin{aligned}
 \|P - \bar{P}\|_v &= \sum_{k \geq 0} \beta^k |P_{0k} - \bar{P}_{0k}| \\
 &= (1 - q) \sum_{k \geq 0} \beta^k \left| \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} dE(x) - \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} dB(x) \right| \\
 &\leq \sum_{k \geq 0} \beta^k \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} |B - E|(dx) \\
 &\leq \int_0^\infty e^{-\lambda x} \sum_{k \geq 0} \frac{(\lambda x \beta)^k}{k!} |B - E|(dx) \\
 &\leq \int_0^\infty e^{\lambda(\beta-1)x} |B - E|(dx) \\
 &\leq \beta W.
 \end{aligned}$$

(ii) For $k \geq 1$:

$$\begin{aligned}
 \|P - \bar{P}\|_v &= \sup_{k \geq 1} \frac{1}{\beta^k} \sum_{j \geq 0} \beta^j |P_{kj} - \bar{P}_{kj}| \\
 &\leq \sup_{k \geq 1} \frac{1}{\beta} \sum_{j \geq k-1} \beta^{j-k+1} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-k+1}}{(j-k+1)!} |B - E|(dx) \\
 &\leq \frac{1}{\beta} \int_0^\infty e^{-\lambda x} \sum_{j \geq 0} \frac{(\lambda \beta x)^j}{j!} |B - E|(dx) \\
 &= \frac{1}{\beta} \int_0^\infty e^{\lambda(\beta-1)x} |B - E|(dx).
 \end{aligned}$$

Then, we have:

$$\|P - \bar{P}\|_v \leq \frac{1}{\beta} \int_0^\infty e^{\lambda(\beta-1)x} |B - E|(dx), \quad (25)$$

now:

$$\int_0^\infty e^{\lambda(\beta-1)x} |B - E|(dx) < \beta W. \quad (26)$$

From which

$$\|P - \bar{P}\|_v \leq W, \quad (27)$$

according to the Lemma (4.1), we will have:

$$\|P - \bar{P}\|_v < \beta_0 W. \quad (28)$$

4.2. STABILITY INEQUALITIES

To be able to apply the Theorem (3.2), let us use the Lemma below for estimate $\|\bar{\pi}\|_v$.

Lemma 4.2. [17] *Let π (respectively $\bar{\pi}$) be the stationary distribution of the embedded chain of the $M/G/1$ queueing system with breakdowns and repairs (respectively of the classical $M/G/1$ system). Then, for all $1 < \beta < \beta_0$, we have*

$$\|\bar{\pi}\|_v \leq c_0 \quad (29)$$

where c_0 is given by

$$c_0 = \frac{(1 - \lambda m)(\beta - 1)}{1 - \rho} \rho \quad (30)$$

and $m = \mathbb{E}(\xi)$, (ξ is the service time).

Let us $c = 1 + \|\mathbf{1}\|_v \|\bar{\pi}\|_v$. Let us use the definition of $\|\cdot\|_v$ in η ,

$\|\mathbf{1}\|_v = \sup_{k \geq 0} \frac{1}{\beta^k} = 1$. Then

$$c = 1 + \|\bar{\pi}\|_v. \quad (31)$$

Theorem 4.2. Let π (respectively $\bar{\pi}$) be the stationary distribution of the embedded Markov chain in the $M/G/1$ queueing system with breakdowns and repairs (respectively of the classical $M/G/1$ system), then for all $1 < \beta < \beta_0$

$$\|\pi - \bar{\pi}\|_v \leq \beta_0 W c_0 c (1 - \rho - c \beta_0 W)^{-1}, \quad (32)$$

where

$$c_0 = \frac{(1 - \lambda m)(\beta - 1)}{1 - \rho} \rho, c = 1 + \|\bar{\pi}\|_v. \quad (33)$$

Proof. According to Theorem (3.2), we have

$$R_0 = (I - T)^{-1} = \sum_{t \geq 0} T^t, \quad (34)$$

$$\|R_0\|_v = \frac{1}{\|I - T\|_v}. \quad (35)$$

Since $\|T\|_v \leq \rho$ we have $\|R_0\|_v \leq \frac{1}{1-\rho}$.

Hence

$$\|R_0(I - T)\|_v \leq \|R_0\|_v(1 + \|\bar{\pi}\|_v) \leq \frac{1 + \|\bar{\pi}\|_v}{1 - \rho}. \quad (36)$$

Using Corollary (3.2), we have

$$\|\pi - \bar{\pi}\|_v \leq \|\bar{\pi}\|_v \|\Delta\|_v \|R_0(I - T)\|_v \frac{1}{1 - \|\Delta\|_v \|R_0(I - T)\|_v} \quad (37)$$

$$\leq \|\bar{\pi}\|_v \|\Delta\|_v \frac{1 + \|\bar{\pi}\|_v}{1 - \rho} \frac{1}{1 - \|\Delta\|_v \frac{1 + \|\bar{\pi}\|_v}{1 - \rho}} \quad (38)$$

$$\leq c_0 c \|\Delta\|_v (1 - \rho \|\Delta\|_v c)^{-1}, \quad (39)$$

Now, we have $\|\Delta\|_v \leq \beta_0 W$.

From which

$$\|\pi - \bar{\pi}\|_v \leq \beta_0 W c_0 c (1 - \rho - c \beta_0 W)^{-1}. \quad (40)$$

CONCLUSION

In this work, we use the approximation conditions [17] for which ones it will be possible to approximate the characteristics of the $M/G/1$ queueing system with breakdowns and repairs by the one of the classical $M/G/1$ queueing system. We obtain stability inequalities with an exact computation of the constants. These theoretical results allows to estimate numerically the error due to this approximation.

REFERENCES

- [1] Thiruvengadam, K. (1963) *Queueing with breakdowns*. Operations Research, 11, 62–71.
- [2] Avi-Itzhak, B. and Naor, P. (1963) *Some queueing problems with the service station subject to breakdown*. Operations Research, 11(3), 303–320.
- [3] Li, W., Shi, D. and Chao, X. (1997) *Reliability analysis of $M/G/1$ queueing systems with server breakdowns and vacations*. Journal of Applied Probability, 34, 546–555.
- [4] Tang, Y. H. (1997) *A single-server $M/G/1$ queueing system subject to breakdowns-some reliability and queueing problem*. Microelectronics and Reliability, 37, 315–321.
- [5] Wang, K. -H., Kao, H. -T. and Chen, G. (2004) *Optimal management of a removable and non-reliable server in an infinite and a finite $M/H_k/1$ queueing system*. Quality Technology and Quantitative Management, 1(2), 325–339.
- [6] Pearn, W. L., Ke, J. -C. and Chang, Y. C. (2004) *Sensitivity analysis of the optimal management policy for a queueing system with a removable and non-reliable server*. Computers and Industrial Engineering, 46, 87–99.
- [7] Wang, J. (2004) *An $M/G/1$ queue with second optional service and server breakdowns*. Computers & Mathematics with Applications, 47, 1713–1723.

- [8] Wang, K. -H., Wang, T. -Y. and Pearn, W. -L. (2005) *Maximum entropy analysis to the N policy $M/G/1$ queueing system with server breakdowns and general startup times*. Applied Mathematics and Computation, 165, 45–61.
- [9] Lam, Y., Zhang, Y. L. and Liu, Q. (2006) *A geometric process model for $M/M/1$ queueing system with a repairable service station*. European Journal of Operational Research, 168, 100–121.
- [10] Ke, J. -C. and Lin, C. -H. (to appear) *Fuzzy analysis of queueing systems with an unreliable server: A nonlinear programming approach*. Applied Mathematics and Computation.
- [11] Baccelli, F. (1980) *Files d'attente avec pannes et applications à la modélisation de systèmes informatiques*. Thèse de Doctorat de 3^{ème} cycle, Université Paris-Sud centre d'Orsay.
- [12] Aïssani, D. and Kartashov, N. V. (1984) *Strong stability of an imbedded Markov chain in an $M/G/1$ system*. Theor. Probability and Math. Statist., 29, 1–5.
- [13] Kartashov, N. V. (1996) *Strong stable Markov chains*. VSP/TBiMC, Utrecht/Kiev.
- [14] Aïssani, D. and Kartashov, N. V. (1983) *Ergodicity and stability of Markov chains with respect to operator topology in the space of transition kernels*. Dokl. Akad. Nauk Ukr. SSR, (ser. A) 11, 3–5.
- [15] Aïssani, D. (1991) *Application of the operators method to obtain inequalities of stability in an $M_2/G_2/1$ system with a relative priority*. Annales Maghrébines de l'Ingénieur (Numéro Hors série 2) 790–795.
- [16] Gross, D. and Harris, C. M. (1974) *Fundamentals of queueing theory*. John Wiley & Sons.
- [17] Berdjoudj, L. and Aïssani, D. (2003) *Strong stability in retrial queues*. Theor. Probability and Math. Statist., 68, 11–17.
- [18] Kartashov, N. V. (1981) *Strong stability of Markov chains*. Proceedings of the All-Union Seminar VNISSI (Stability Problems for Stochastic Models), 30, 54–59.
- [19] Bouallouche-Medjkoune, L. and Aïssani, D. (2006) *Performance analysis approximation in a queueing system of type $M/G/1$* . Mathematical Methods of Operations Research (ZOR), 63(2), 341–356.