Research Article

Solution of Telegraph Equation by Elzaki-Laplace Transform

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Abstract

In this paper we have studied the combination of Elzaki and Laplace transforms to produce a degenerative new double integral transform, this new transform called the Elzaki-Laplace transform (ELT), and then applied it to some partial differential equations to find its exact solution.

Keywords: Elzaki-Laplace transform, integral transforms, double integral transform, partial differential equations.

1- Introduction

Double integral transformations, their properties and theories are still new and under study [6, 9]. The previous studies dealt with a few parts of them, such as definitions, simple theories and solution of ordinary and partial differential equations [4, 8]. Moreover, some researchers have dealt with these transformations and combined them with various mathematical methods such as differential transformation method, homotopy perturbation method, Adomian decomposition method and variational iteration method [1-5, 10, 11] to solve the nonlinear fractional differential equations.

Integral transform has played an important role in solving differential equations and integral equation, through transform these equations to algebraic equation. In addition, there are many integral transforms that have been used in many of the solution to the problems under initial conditions, which are difficult to resolve the classic ways like Laplace, Elzaki, Temimi and Novel …etc. [12 -15]. Laplace transform is introduced by Pierre –Simon Laplace that became one of the famed transform in mathematics, engineering and physics [13, 15]

In the resent years, Tarig Elzaki [9] have applied the double Elzaki transform to solve general linear telegraph and partial differential equations, and study the convergence of triple Elzaki transform. Ranjit R. Dhundel, G. L. Waghmare [16] are solving partial integro-differential equations using double Laplace transform method,
In this paper we solve telegraph equation using Elzaki - Laplace transform method, we directly convert telegraph equation into an algebraic equation instead of converting to ODE. Solving this algebraic equation & applying double inverse Elzaki - Laplace transform we obtain the exact solution. This method is illustrated by giving two examples.

1.1 Basic Definitions and Theorems

Definition (2.1):
The Laplace and Elzaki transforms of continuous (almost piecewise continuous) functions $g(\lambda)$, $g(\tau)$ in $[0,\infty)$ are defined as:

$$
\ell[g(\lambda)] = \int_0^{\infty} g(\lambda) e^{-\mu \lambda} d\lambda, \quad (1)
$$

$$
e[e^\lambda g(\tau)] = \int_0^{\lambda} e^{-\mu \lambda} d\lambda, \quad (2)
$$

Definition (2.2): [6]
The Elzaki-Laplace transform (ELT) of the function $g(\tau, \lambda)$, is defined by the double integral,

$$
G(\theta, \rho) = \ell[e\lambda g(\tau, \lambda)] = \ell[e^{\lambda \tau} g(\tau, \lambda)] = \theta \int_0^{\infty} g(\tau, \lambda) e^{-\mu \lambda} d\lambda d\lambda \quad (3)
$$

Where $G(\theta, \rho)$ is the Elzaki-Laplace transform of the function $g(\tau, \lambda)$,

This is the linear transformation.

The inverse (ELT), is defined by the formula,

$$
g(\tau, \lambda) = (e\lambda)^{-1}[G(\theta, \rho)] = \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta e^{-\mu \lambda} e^{\lambda \tau} d\theta d\rho \quad (4)
$$

Elzaki-Laplace Transform (ELT) of Some Functions

If $g(\tau, \lambda) = \tau^a \lambda^b$, $\tau, \lambda > 0$, then:

$$
G(\theta, \rho) = e\lambda [\tau^a \lambda^b] = \theta \int_0^{\infty} \tau^a \lambda^b e^{-\mu \lambda} d\lambda d\lambda = \int_0^{\infty} \theta e^{-\mu \lambda} d\lambda \int_0^{\infty} \lambda^b e^{-\mu \lambda} d\lambda = \frac{\Gamma(b+1)}{\rho^{b+1}} \theta^{a+2} \Gamma(a+1)
$$

For especial cases: $e\lambda [1] = \frac{\theta^a}{\rho}$,
And if \( a, b \) are positive integer numbers, then:

\[
\varepsilon \left[ \tau^{a} \lambda^{b} \right] = \frac{a! b!}{\rho^{a+b}} \Theta^{a+b+2}
\]

If \( g(\tau, \lambda) = e^{\tau + b \lambda} \), then:

\[
G(\theta, \rho) = \varepsilon \left[ e^{\tau + b \lambda} \right] = \theta \int_{0}^{\infty} e^{\frac{\tau}{\theta} + b \lambda} d\tau d\lambda = \int_{0}^{\infty} \theta e^{-\left(\frac{1}{\theta} - a\right)\lambda} d\lambda \left( e^{-\rho \lambda} - e^{-(\rho - b)\lambda} \right)
\]

In a similar way, we can prove that:

\[
\varepsilon \left[ \sin(\tau + b \lambda) \right] = \frac{b \theta^{2} + \rho a \theta^{3}}{(\rho^{2} + b^{2})(1 + a^{2} \theta^{2})}, \quad \varepsilon \left[ \cos(\tau + b \lambda) \right] = \frac{\rho \theta^{2} - ab \theta^{3}}{(\rho^{2} + b^{2})(1 + a^{2} \theta^{2})}
\]

\[
\varepsilon \left[ \sinh(\tau + b \lambda) \right] = \frac{b \theta^{2} + \rho a \theta^{3}}{(\rho^{2} - b^{2})(1 - a^{2} \theta^{2})}, \quad \varepsilon \left[ \cosh(\tau + b \lambda) \right] = \frac{\rho \theta^{2} - ab \theta^{3}}{(\rho^{2} - b^{2})(1 - a^{2} \theta^{2})}
\]

**Theorem (2.3):** If a function \( g(\tau, \lambda) \), continuous in finite interval \((0, X)\) and \((0, T)\), is of exponential order \( e^{\tau + b \lambda} \) and \( \varepsilon \left[ g(\tau, \lambda) \right] = G(\theta, \rho) \) then:

\[
\varepsilon \left[ \frac{\partial g(\tau, \lambda)}{\partial \lambda} \right] = \rho G(\theta, \rho) - \varepsilon \left[ g(\tau, 0) \right]
\]

\[
\varepsilon \left[ \frac{\partial g(\tau, \lambda)}{\partial \tau} \right] = \frac{1}{\theta} G(\theta, \rho) - \theta \varepsilon \left[ g(0, \lambda) \right]
\]

**Proof:**

\[
\varepsilon \left[ \frac{\partial g(\tau, \lambda)}{\partial \lambda} \right] = \theta \int_{0}^{\infty} \frac{\partial g(\tau, \lambda)}{\partial \lambda} e^{-\frac{\tau}{\theta} - b \lambda} d\tau d\lambda = \theta \int_{0}^{\infty} e^{-\frac{\tau}{\theta}} d\tau \int_{0}^{\infty} e^{-b \lambda} \frac{\partial g(\tau, \lambda)}{\partial \lambda} d\lambda
\]

Integrating the second integral by parts to find:

\[
\varepsilon \left[ \frac{\partial g(\tau, \lambda)}{\partial \lambda} \right] = \theta \left[ e^{-\frac{\tau}{\theta}} g(\tau, \lambda) \right]_{\frac{\tau}{\theta} = 0} + \rho \int_{0}^{\infty} e^{-b \lambda} g(\tau, \lambda) d\lambda = \rho G(\theta, \rho) - \varepsilon \left[ g(\tau, 0) \right]
\]

By the same method, we can prove (7).

Similarly we write:
2 Application of the Elzaki-Laplace Transform (ELT) to the Telegraph Equation

In this section, we will solve the telegraph equation in the following steps:

(i) Take the ELT, of the Telegraph equations.

(ii) Take the single Elzaki and Laplace transforms of the given conditions.

(iii) Set up an algebraic equation using (i) and (ii).

(iv) To determine the solution, take the inverse of ELT.

We'll start by looking at the general telegraph equation:

\[ q_{rr} + aq_r + bq = c^2 q_{\lambda \lambda} \]  \hspace{1cm} (9)

With the conditions:

\[ q(\tau,0) = f_1(\tau), \quad q_\lambda(\tau,0) = g_1(\tau), \quad q(0,\lambda) = f_2(\lambda), \quad q_\lambda(0,\lambda) = g_2(\lambda) \]  \hspace{1cm} (10)

**Solution:**

Take the ELT of equation (9) and the single Elzaki and Laplace transform of conditions (10), to get:

\[
\frac{1}{\theta^2} Q(\theta, \rho) - \ell[q(0,\lambda)] - \theta \ell[q_\lambda(0,\lambda)] + \frac{a}{\theta} Q(\theta, \rho) - a \theta \ell[q(0,\lambda)] + b Q(\theta, \rho)
= c^2 \left\{ \rho^2 Q(\theta, \rho) - \rho \epsilon[q(\tau,0)] - \epsilon[q_\lambda(\tau,0)] \right\}
\]  \hspace{1cm} (11)

Where:

\[
\epsilon[q(\tau,\lambda)] = Q(\theta, \rho), \quad \ell[q(0,\lambda)] = \ell[f_2(\lambda)] = F_2(\rho), \quad \ell[q_\lambda(0,\lambda)] = \ell[g_2(\lambda)] = G_2(\rho), \quad \\
\epsilon[q(\tau,0)] = \epsilon[f_1(\tau)] = F_1(\theta), \quad \epsilon[q_\lambda(\tau,0)] = \epsilon[g_1(\tau)] = G_1(\theta)
\]

Then Eq. (11) becomes:
\[
\frac{1}{\partial^2} Q(\theta, \rho) - F_2(\rho) - \theta G_2(\rho) + a \frac{\partial}{\partial \theta} Q(\theta, \rho) - a \theta F_1(\rho) + b Q(\theta, \rho) \\
= c^2 \left\{ \rho^2 Q(\theta, \rho) - \rho F_1(\theta) - G_1(\theta) \right\}, \quad \Rightarrow \\
Q(\theta, \rho) = \left( \frac{a \theta^3 + \theta^2}{1 + a \theta + b \theta^2 - c^2 \theta^2 \rho^2} \right) F_2(\rho) + \theta^2 G_2(\rho) - c^2 \theta^2 \rho F_1(\theta) - c^2 \theta^2 G_1(\theta) = H(\theta, \rho)
\]

And if the inverse of ELT exists, then the solution comes soon after the inversion of Eq. (12),
\[
q(\tau, \lambda) = (\varepsilon \ell)^{-1} [H(\theta, \rho)]
\]

**Example 1**

Let us consider the linear telegraph equation,
\[
q_{\lambda\lambda} = q_{\tau\tau} + q + q
\]

With the conditions:
\[
q(\tau, 0) = e^{-\tau}, \quad q_{\tau}(\tau, 0) = e^{-\tau}, \quad q(0, \lambda) = e^\lambda, \quad q_{\tau}(0, \lambda) = -e^\lambda
\]

**Solution**

Take the ELT from both sides of Eq. (13), and use of conditions (13), to find:
\[
\left[ \rho^2 - 1 - 1 - \theta - 1 \right] Q(\theta, \rho) = \frac{\rho^2 \theta^2}{1 + \theta} + \frac{\theta^2}{1 + \theta} - \frac{1}{\rho - 1} + \frac{\theta}{\rho - 1} - \frac{\theta}{\rho - 1} \\
\left[ \rho^2 \theta^2 - 1 - \theta - \theta^2 \right] Q(\theta, \rho) = \frac{\rho^2 \theta^2 - 1 - \theta - \theta^2}{(1 + \theta)(\rho - 1)}, \quad \Rightarrow \quad Q(\theta, \rho) = \frac{\theta^2}{(1 + \theta)(\rho - 1)}
\]

By taking the inverse of ELT, we find the exact solution of Eq. (13), in the form:
\[
q(\tau, \lambda) = e^{-\tau} e^\lambda = e^{\lambda - \tau}
\]

**Example 2**

Consider the following non-homogeneous Telegraph equation,
\[
q_{\lambda\lambda} - q_{\tau\tau} - q_{\tau} + 2q = e^{\lambda - 2\tau}
\]

With the conditions:
\[
q(\tau, 0) = e^{-2\tau}, \quad q_{\tau}(\tau, 0) = e^{-2\tau}, \quad q(0, \lambda) = e^\lambda, \quad q_{\tau}(0, \lambda) = -2e^\lambda
\]

**Solution**

By using the same steps as in example 1, we find that:
\[
\left[ \rho^2 - \frac{1}{\theta^2} - \frac{1}{\theta} + 2 \right] Q(\theta, \rho) = \frac{\rho \theta^2}{1 + 2 \theta} + \frac{\theta^2}{1 + 2 \theta} - \frac{1}{\rho - 1} + \frac{2 \theta}{\rho - 1} - \frac{\theta}{\rho - 1}
\]

Simplify to find:

\[
Q(\theta, \rho) = \frac{\theta^2}{(1 + 2 \theta)(\rho - 1)}
\]

Take the inverse of ELT to find the exact solution of Eqs. (15), (16) in the form:

\[
q(\tau, \lambda) = e^{-\tau} e^{\lambda} = e^{\lambda - 2\tau}
\]

3 Conclusion

In this study, we combined the Elzaki and Laplace transforms to produce a new double transform called Elzaki-Laplace transform (ELT), and then applied it to the telegraph equation to find the exact solution of it. The result was excellent where this equation was solved simply and easily. Since this transform is new, we will apply it to other partial differential equations and integral equations in future studies.

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

The authors read and agreed the final manuscript.

References


