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BRAHMAGUPTA AND THE CONCEPT OF ZERO

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Abstract.

This paper attempts to reconstruct the possible reasoning process that led the great Indian mathematician Brahmagupta in 628 A.D. to the formulation of two controversial rules for arithmetic, involving the number zero; rules which contradict modern arithmetic principles. Is it possible to explain these rules in some logical manner? This paper outlines a possible explanation of the issue based on similar reasoning. One may ask, why is the concept of zero so important? “From counting to calculating, from estimating the odds to knowing exactly . . . all of their parts swing on the smallest of pivots, zero” Kaplan [12]. Today’s technology would simply be impossible, from the smallest electronic device to space technology, engineering, mathematics and physics. If it were possible to erase the existence of zero from the annals of human achievement, we would be thrown back into the ancient times. Humanity owes a great debt of gratitude to the original inventor of zero - Brahmagupta, as well as to the Indian culture.

In the history of culture the discovery of zero will always stand out as one of the greatest single achievements of the human race.

Tobias Danzig

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1. Historical note on Brahmagupta

Brahmagupta was born in the city of Bhimmal, lived from 598 to 670 A.D. He was a great Indian mathematician and astronomer who wrote several books. At the age of 30, in the year 628 A.D. he wrote a treatise called Brahmasphutasiddhanta (translated as “The opening of the Universe”), it was one of his early works. It is speculated that Brahmagupta revised the text of Siddhanta that he received; nevertheless, he also brought a great deal of new material into it.

A large part of this treatise concerns astronomy, but it also includes mathematics, trigonometry, algorithms and algebra. While further developing the already known notion of the placeholder in numbers, Brahmagupta actually realized that mathematics needs a new number. Consequently, in his work he introduced one of the fundamental discoveries in mathematics - the concept of the number zero. Brahmagupta called the new number sūnya, which in Sanskrit means "void" or "empty".

Brahmagupta was the first mathematician who conceived the concept of zero, and defined its arithmetical properties, respectfully he is called the father of zero. He argued that sūnya (zero) is a number that represents nothing, and stated “sūnya is actually the result of subtracting a number from itself”, a concept which nowadays everyone understands, in his time however it was a completely novel idea.

Brahmagupta also criticized contemporary Indian astronomers and mathematicians. The core of the dispute was related to the differing views and concepts; in fact he devoted several chapters of Brahmasphutasiddhanta to criticize their mathematical concepts. After Arabs conquered a part of India, Brahmagupta’s works were translated by the astronomer Muhammad al-Fazari into Arabic, thus promoting the advance and implementation of the Indian base ten number system, Hoffman [11].

2. The brief history of zero

2.1. The placeholder concept.

Already the ancient mathematicians of Babylonia, Greece and China recognized that a placeholder is necessary in order for the numbers to display and hold the correct values. The placeholder concept was invented independently in a number of civilizations around the world, says Dr. Annette van der Hoek, the lead researcher at the Zero Project.

The Sumerians were the first people in the world to develop a counting system, they developed their counting system around 4,000 to 5,000 years ago. The Sumerian counting system, similarly to the modern decimal system, was positional; this means that the value of a symbol depended on its position in the sequence of other symbols.

Around 300 B.C. the Sumerian system passed from the Akkadian Empire to the Babylonians who were using base 60 counting system. It was the Babylonians, Harvard mathematics professor Robert Kaplan maintains, that implemented a symbol that obviously was a placeholder. The placeholder facilitated clear discernment of 10 from 100 for example. Initially, the Babylonians left an empty space in their cuneiform number, but when that became confusing, they implemented in its place
double angled wedges. Babylonians however, never developed the idea of a zero as a number.

The Mayans, independently invented zero as a placeholder around 350 A.D. implementing it in their calendar system. Kaplan says that despite being highly skilled mathematicians, the Mayans never used zero in equations. The Mayan invention of zero however, is the "most striking example of the zero being devised wholly from scratch".

In 1891 archeologist A. Leclere discovered an inscription in Cambodia, containing the “dot” as a placeholder for zero. This inscription, later classified as K-127 by G. Coedes, has been lost for decades, following the Khmer Rouge destruction of cultural artifacts. Later the stone containing the inscription has been re-discovered by A. Aczel. “Archeologists date this phrase to 687 AD, in pre-Angkor Cambodia … Cambodia has many inscriptions with the zero symbol, but this one K-127 is the oldest one” Chéa Socheat [21]. It is the oldest record of reference to zero in Cambodia. A symbolic placeholder however, cannot be equated with zero itself. Because it lacks the most important qualities of zero - precise arithmetical properties.

The Bakhshali manuscript is an early ancient mathematical document discovered in a field in 1881 by a farmer, near the village of Bakhshali in the vicinity of Peshawar, at that time in British India. The manuscript is written in ink on a birch bark, which was usual for manuscripts in North-Western India. It displays some Sanskrit numerals, in which in the place of zero is placed a small dot. It is a compendium of rules and examples and presents verified solutions. The content of the manuscript covers square root computations to extraordinary accuracy, fractions, quadratic equations, simultaneous linear equations, indeterminable equations of the second degree and many other problems. News Staff [19].
The fragments comprising its pages, were once part of a more extensive document. With a considerable part of it missing and the uncertainty of the correct order of the pages, we may only try to guess its significance. Attempts to assess the manuscript were published by G. Kaye of the Indian Government in 1927 and Takao Hayashi in 1995. Hayashi attempted to order the pages and translate them, his translation is considered superior to the 1927 translation by Kaye. The manuscript had been donated in 1902 to the Oxford University Library.

In 2017 researchers at Oxford University carried out a comprehensive research of the Bakhshali manuscript, aiming to carbon date the document. The manuscript has been dated to be from ca 300 to 400 A.D., this carbon dating however resulted in a strong criticism by a group of researchers cf. Plofker et al [16]. Independently of that, a much earlier thorough research done by Channabasappa [6] as well as Datta [8], based on the results contained in the Bakhshali manuscript, yielded the approximate origin of the manuscript to be from ca 200 to 400 A.D. Channabasappa derived from the Bakhshali manuscript an iterative method to calculate square roots, which is significantly faster than the Newton’s method.

According to Casselman [5], the Bakhshali manuscript does not tell us much new, what has not been seen in other Indian mathematical manuscripts. It does however present “Extraordinarily accurate approximations to the square roots of integers that are not perfect squares…My own personal belief is that the original work from which the Bakhshali manuscript originates was very close to the invention of the full decimal place system of arithmetic”.

2.2. The birth and advance of zero.

The Indian Hindu culture had a positional number system implementing the base ten. In these early counting systems, the placeholder has not been considered a number with its own properties. Comprehension of zero’s concept significance and its properties, developed first during the seventh century A.D. in India.

In the year 628 A.D. Brahmagupta wrote a treatise called Brahmasphutasidhantha (translated as “The opening of the Universe”). Brahmagupta actually realized, that mathematics needs a new number. Consequently, in the treatise he introduced one of the fundamental discoveries in mathematics - the concept of the number zero. Brahmagupta called the new number sunya, which in Sanskrit means “void” or “empty”.

Hindu mathematicians Aryabhata (born in 476 A.D.) and Brahmagupta (born in 598 A.D.) are believed to be the first mathematicians who formally described the modern decimal number system.

“The creation of zero as a number in its own right, which evolved from the placeholder dot symbol found in the Bakhshali manuscript, was one of the greatest breakthroughs in the history of mathematics. We know that it was as early as the 3rd Century that mathematicians in India planted the seed of the idea that would later become so fundamental to the modern World”, said Marcus du Sautoy, Professor of Mathematics at University of Oxford, the lead researcher.

The oldest written record of zero as a number in its own right is located in the Chaturbhuj Temple in Gwalior, India; please refer to Fig. 2, where a 9th century carving on the wall displays the number 270, Ward [24].
Figure 2. The LH Figure shows the oldest written record of zero as a number in its own right, carved in stone, from the Chaturbhuj Temple in Gwalior, India. Standage [22] The RH Figure shows the Indian Sanskrit numbers and the placeholder zero.

Comprehension of zero’s significance and its properties spread also beyond India. The concept of zero initially made its way to China and then back to the Middle East, where it was taken up by the mathematician Mohammed ibn-Musa al-Khowarizmi around 773, becoming part of the Arabic number system based upon the Indian system, Hoffman [11].

It was Al-Khowarizmi who first synthesized Indian arithmetic and showed how zero could function in algebraic equations, and by the ninth century the zero had entered the Arabic numeral system in a form resembling the oval shape we use today. The Arabs called this circle "sifr," or "empty"; the method for multiplying...
and dividing numbers, which nowadays is known as an algorithm, is a derived from the title of the Latin version of his book.

Zero is undoubtedly one of the greatest achievements in mathematics. The actual term zero was coined in Italy. In the decimal number system over $\mathbb{R}$, zero separates the negative and the positive numbers, itself being neither negative nor positive. Zero facilitated the rise of the modern mathematics as we know it. "So commonplace has zero become, that few if any, realize its astounding role in the lives of every single person in the world" said Peter Gobets, the secretary of the ZeroOrigIndia Foundation, or the Zero Project.

Further he says that "the Indian [concept of] zero, widely seen as one of the greatest innovations in human history, is the cornerstone of modern mathematics and physics, plus the spin-off technology". Hence, it is behind every modern achievement, as it lies at the very foundation of the modern science and technology. It is omnipresent today, its applications stretch from mathematics to engineering, in fact zero permeates our everyday lives.

2.3. The banning of zero.

The concept of zero spread to Europe ca 1200, popularized in the work of Leonardo Fibonacci of Pisa Italy - Liber Abaci, Matson [15]. Fibonacci, used it to evaluate equations. Fibonacci's equations involving zero were favored among merchants, who used it to balance their books.

Soon afterward however, zero was banned in Europe because, as the proponents of such a move maintained, zero promotes the possibility of fraud. They said that zero can easily be altered to become 9, nothing stops one from adding a few zeroes at the end of a number etc. As a result, zero and the Arabic numerals were banned in 1299. It took approximately 200 years before the Arabic numerals and zero were finally accepted in Europe, Fry [9].

Since then, the concept of "nothing" has continued to play a vital role in the development of everything from science and engineering to technology. According to Prof. Brian Rotman "Between the tenth and the thirteenth century the sign [Zero] stayed within the confines of Arab culture, resisted by Christian Europe, and dismissed by those whose function it was to handle numbers as an incomprehensible and unnecessary symbol . . . But, as will be obvious by now, the mathematical infinite was the fruit of the mathematical nothing: it is only by virtue of zero that infinity comes to be significant in mathematics", Rotman [18].

Although zero and nothing point to a state of absence (in general), they are not equivalent. Therefore the question arises: what is the difference between zero and nothing?

- Zero is a number which has very precise arithmetical properties, it is the result of arithmetical/counting process. Nothing is just an abstract concept which has no arithmetical properties, it is uncountable - it either is true (i.e. state of complete absence of some quantity or magnitude) or false (i.e. the presence of some quantity or magnitude is obvious).

- Zero is the only number which is neither positive nor negative at the same time, it has an exact numerical value and position on the number line. Nothing neither has a position on the number line nor exact arithmetical value.
3. Brahmagupta’s Concept of zero expounded

Brahmagupta’s ten rules for zero, Bentley [4]:

(1) A debt minus zero is a debt.
(2) A fortune minus zero is a fortune.
(3) Zero minus zero is zero.
(4) A debt subtracted from zero is a fortune.
(5) A fortune subtracted from zero is a debt.
(6) The product of zero and a debt or fortune is zero.
(7) The product of zero and zero is zero.
(8) Positive or negative numbers, when divided by zero result in a fraction with the zero in the denominator.
(9) Zero divided by a negative or a positive number is either zero, or is expressed as a fraction with zero as numerator and the finite quantity in the denominator.
(10) Zero divided by zero is zero.

For the sake of simplicity, we limit the discussion to the set of rational numbers only. Denoting a fortune by a positive natural number and the debt by a negative natural number, clearly shows that eight of the Brahmagupta’s rules for zero are in agreement with the rules of modern mathematics.

Definition 3.1.

\[ \text{fortune} = x \in \mathbb{N} \mid x > 0 \]

Definition 3.2.

\[ \text{debt} = y \in \mathbb{N} \mid y < 0, \text{ and } y = -x \]

Then, eight of the above rules (1 through 7 and 9) can be written as:

\[
\begin{align*}
 y - 0 & = y \\
x - 0 & = x \\
0 - 0 & = 0 \\
0 - y & = x \\
0 - x & = y \\
y \times 0 & = 0 \\
x \times 0 & = 0 \\
0 \times 0 & = 0 \\
\frac{0}{x} & = \frac{0}{y} = 0
\end{align*}
\]

There is nothing unusual about these statements, just standard arithmetic. Rules 8 and 10 however, do not conform with the rules of standard arithmetic, in fact these operations are undefined.

In order to figure out and maybe understand Brahmagupta’s logic, we need to step out of the box and try to think unconventionally. **Let’s put the modern standard mathematical rules concerning division, aside for a moment.**

We can think of the division process in terms of sharing or distributing some items. The discussion revolves about one unspecified item shared among some participants (w.l.o.g. assuming that the item is infinitely divisible).
The most that a person can obtain is the whole item. If we are to share the item among say 2 people, each person obtains: (1 person) \( \frac{1}{2} \) shares of the item, considering 3 people, each person obtains: (1 person) \( \frac{1}{3} \) shares etc. Generalizing, an item is split into equal shares among \( x \in \mathbb{N} \) individuals and as the number of individuals \( x \) increases unboundedly, the share size inevitably tends to:

\[
\lim_{x \to \infty} \left( \frac{1 \text{ item}}{x \text{ individuals}} \right) \to 0
\]

What about going in the other direction \( \forall x \in \mathbb{N} \mid 0 < \frac{1}{x} \leq 1 \)? Going to the left of 1 we have:

\[
\frac{1 \text{ item}}{1 \text{ individual}} = 1, \quad \frac{1 \text{ item}}{\frac{1}{2} \text{ individual}} = 2(1) = 2 \text{ items/individual}, \ldots,
\]

\[
\frac{1 \text{ item}}{\frac{1}{10^6} \text{ individual}} = 10^6(1) = 10^6 \text{ items/individual}
\]

It is obvious that, the graph of such a function increases rapidly as the ratio of person per share gets closer and closer to zero,

\[
\lim_{x \to \infty} \left( \frac{1 \text{ item}}{\frac{1}{x} \text{ individual}} \right) \to \infty
\]

Hence, going to the left of 1 implies multiplication by rapidly growing numbers, while going to the right of 1 represents a division by increasing numbers. If there is nothing to share,

\[
\frac{0 \text{ item}}{x \text{ individuals}} \Rightarrow \text{ an individual gets} \left( \frac{0}{x} \right) \Rightarrow \text{ an individual clearly gets 0 shares}
\]

Going one step further, an item exists to be shared if desired; however, for a reason that we ignore, the item will not be shared. As there are no participants to share, thus \( x = 0 \) in this case. Clearly in this situation, there exists one item at our disposal and this will not change. Hence,

\[
\frac{1 \text{ item}}{0 \text{ individual}} \Rightarrow \text{ the item remains left over intact}
\]

To make this work, we would have to define that in such a case the fraction reduces to (due to the lack of participants and unchanged status of the item):

\[
\frac{1 \text{ item}}{0 \text{ individual}} \Rightarrow 1
\]

Generalizing

\[
\frac{n \Rightarrow n}{0 \text{ for } n \in \mathbb{N}}
\]

The next step would be that there exists no item to share and that there is nobody to share it with either. Thus we have zero items and zero individuals to share it among. Physically and in full agreement with common sense, this must result in nothing or zero in the end:

\[
\frac{0 \text{ items}}{0 \text{ individuals}} \Rightarrow \text{ there never was anything and nothing is remaining}
\]
Hence,
\[(3.10) \quad \frac{0 \text{ items}}{0 \text{ individuals}} \Rightarrow 0\]

Seems logical and is certainly based on layman’s common everyday experience, as we can not create anything out of complete void and there is no one to share it with either.

Summarizing the above then, in accordance with definition 3.1,
\[(3.11) \quad \lim_{x \to 0} \left( \frac{1 \text{ item}}{\frac{1}{x} \text{ individual}} \right) \to \infty \quad \forall x \in \mathbb{N} | 0 < \frac{1}{x} \leq 1\]

and for \(x = 0\),
\[(3.12) \quad \left( \frac{1 \text{ item}}{0 \text{ individuals}} \right) \Rightarrow 1\]

while for both \(x = 0\) as well as \(\text{item } = 0\) we have,
\[(3.13) \quad \left( \frac{0 \text{ items}}{0 \text{ individuals}} \right) \Rightarrow 0\]

This reasoning indicates approximately how Brahmagupta might have been thinking about zero and the rules of arithmetic. Obviously rules 8 and 10 do not conform with the rules of modern arithmetic; it is however interesting to know how Brahmagupta might have conceived those rules and what might have nurtured the ideas in his mind. It is common for ideas to travel on a bumpy road from the day of their conception; later however, they are refined again and again, until eventually we get them in order. Extending the Brahmagupta’s argument to roots:
\[(3.14) \quad \sqrt[d]{a} = a^{\left(\frac{1}{d}\right)}\]

Implementing expression 3.12, in case of \(d = 0\) above we obtain:
\[(3.15) \quad \sqrt[0]{a} = a^{(1)} = \sqrt[0]{a} = a\]

Hence the zero\(^{th}\) root in accordance with the presented theory, precisely equals the first root, which is exactly the number \(a\) itself. Extending the Brahmagupta’s concept to include equation 3.14 and 3.15 shows the implications of accepting such a position.

4. CONCLUSION

The invention of zero by the great Indian mathematician Brahmagupta and its effect on the progress of human civilization can be compared to the effect of the discovery of fire, the invention of wheel or computer. In fact, the computer crucially owes its existence and progress to zero.

All these drastically changed the course of human development. Without one of those, civilizational progress would probably stall.

Teaching about the history of the invention of zero should be universally introduced to schools. It is a long and quite involved story, it would however be of great advantage for every student to be able to appreciate the value of zero and its contribution into everyone’s life.
References