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Authors: Jan Feliksiak[1]

Affiliations: N.A.[1]

Orcid ids: 0000-0002-9388-1470[1]

Contact e-mail: jan.feliksiak1@yahoo.com

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THE CONGRUENT NUMBER PROBLEM

JAN FELIKSIK

ABSTRACT. The congruent number problem is the oldest unsolved major mathematical problem to date. The problem aiming to determine whether or not some given integer n is congruent, which corresponds to a Pythagorean triangle with integer sides, can be settled in a finite number of steps. However, once we permit the triangles to acquire rational values for its sides, the degree of difficulty of the task changes dramatically. In this paper a basis is developed, to produce right Pythagorean triangles with rational sides and integral area in a straightforward manner. Determining whether or not a given natural number n is congruent, is equivalent to a search through an ordered 2D array.

A problem in number theory is as timeless as a true work of art.
David Hilbert

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NOTATION AND GLOBAL DEFINITIONS

A Pythagorean triple $\{a, b, c\}$, is defined as:

$$(0.1) \quad a, b, c \in \mathbb{Q} \mid a, b, c > 0, a^2 + b^2 = c^2 \text{ for some } n \in \mathbb{N} \mid n = \frac{ab}{2}$$

The triangle class criterion (TCC) is defined as the difference between the two longest sides of the right Pythagorean triangle:

Definition 0.1 (Triangle Class Criterion). $d = c - b$

The even/odd sub-sequence of the TCC class criterion values is denoted by *ESS* or *OSS* respectively.

0.1. Integral Pythagorean triples.

The variables s and k are used to define the location of a Pythagorean triple in a two dimensional array. The variable s denotes the row in the array, while the variable k denotes the column of the array.

0.1.1. Infinite sequence of triangles.

We define here the notation implemented in Theorem 2.7 to generate an infinite sequence of triangles having identical square-free area (copies of the first one). Hence, we define:

s_I, k_I : The initial coordinates i.e. values for the row s and the column k in the array, at which the first (of interest) Pythagorean triple is located, which generates a triangle of some given square free area.

s_T, k_T : The sought after - target values of s and k .

I_T : The index of the particular target Pythagorean triple i.e. 2 - for the next, 3 for the following etc., which generates a triangle of identical square free area as the initial one.

0.1.2. Modified parameters for Euclid's sufficiency condition.

Definition 0.2.

$$u = \left(k + \frac{1}{2}\right) \sqrt{2}, \text{ and } v = \left(s - \frac{1}{2}\right) \sqrt{2}, \quad \text{for any } k, s \in \mathbb{N} \mid k \geq s$$

0.2. Rational Pythagorean triples s.t. $c - b = \frac{1}{t} \mid t \in \mathbb{N}$.

Definition 0.3.

$$q \in \mathbb{N} \mid q = \{1, 3, 5, 7\}$$

Definition 0.4.

$$z = \begin{cases} t & \text{for any } t \in \mathbb{N} \mid t > 2 \\ \frac{t}{2} & \text{for any } t \in \mathbb{N} \end{cases} \quad \begin{matrix} \text{s.t.} & t \equiv 1 \pmod{2} \\ \text{s.t.} & t \equiv 0 \pmod{2} \end{matrix}$$

Definition 0.5.

$$w = \begin{cases} \{q-1, q, q+1\} & \text{s.t. } q = \{3, 5\} \text{ and for } t \in \mathbb{N} & \text{s.t. } t \equiv 1 \pmod{2} \\ \{q-1, q, q+1\} & \text{s.t. } q = \{1, 7\} \text{ and for } t \in \mathbb{N} & \text{s.t. } t \neq 4r \mid r \in \mathbb{N} \\ \{q-1\} & \text{s.t. } q = 1 \text{ and for } t \in \mathbb{N} & \text{s.t. } t = 4r \mid r \in \mathbb{N} \\ \{q+1\} & \text{s.t. } q = 7 \text{ and for } t \in \mathbb{N} & \text{s.t. } t = 4r \mid r \in \mathbb{N} \end{cases}$$

Which are used to formulate m :

Definition 0.6.

$$m = z^2 j + \frac{w}{8} (z-1)(z+1) \mid \text{ for } j \in \mathbb{N} \cup \{0\}$$

Remark 1. Subject to (depending on the value of w), j and w are not simultaneously equal to zero in the Definition 0.6 of m .

Finally, the variables k and s :

Definition 0.7.

$$k = \frac{1}{2t} \left(-t + (8m + q) \sqrt{t} \right) \mid, \text{ with } m \text{ as given by the Definition 0.6.}$$

Definition 0.8.

$$s = \frac{1}{2} + \frac{1}{2\sqrt{t}}$$

0.3. **Rational Pythagorean triples s.t.** $c - b = \frac{y}{x} \mid x, y \in \mathbb{N}$.

Definition 0.9.

$$t \in \mathbb{Q} \mid t = \frac{x}{y} \text{ subject to } x, y > 1$$

in accordance with the Definition 0.3 we define the variables w and j :

Definition 0.10.

$$w = \begin{cases} \{q-1, q+1\} & \text{s.t. } q = \{1, 3, 5, 7\} \text{ and for } t \in \mathbb{Q} & \text{s.t. } x \equiv 0 \pmod{2} \\ \{q-1\} & \text{s.t. } q = \{1, 5\} \text{ and for } t \in \mathbb{Q} & \text{s.t. } y \equiv 0 \pmod{2} \\ \{q+1\} & \text{s.t. } q = \{3, 7\} \text{ and for } t \in \mathbb{Q} & \text{s.t. } y \equiv 0 \pmod{2} \\ \{q-1, q, q+1\} & \text{s.t. } q = \{1, 3, 5, 7\} \text{ and for } t \in \mathbb{Q} & \text{s.t. } x, y \equiv 1 \pmod{2} \end{cases}$$

Definition 0.11.

- (1) $j = 2^{s-1} y n \mid n \in \mathbb{N}$, for $y = 2^\alpha$, where:
 - $\alpha = s + 1 \mid s = 0$, or
 - $\alpha = 3s + r \mid r = \{2, 3, 4\}$ with $s \in \mathbb{N} \cup \{0\}$.
- (2) $j = 2^{s-1} y (p_1^{s_1+1} p_2^{s_2+1} \dots p_i^{s_i+1}) n + \nu \mid n \in \mathbb{N}$ for $y = 2^\alpha (p_1^{\beta_1} p_2^{\beta_2} \dots p_i^{\beta_i})$

where: $\beta_i = 3s_i + r \mid s_i \in \mathbb{N} \cup \{0\}$, $r = \{1, 2, 3\}$ and $p_i^{\beta_i}$ are the distinct odd prime factors of y with their respective exponents β_i that occur and ν is a constant which depends on the value of y .
- (3) $j = y (p_1^{s_1+1} p_2^{s_2+1} \dots p_i^{s_i+1}) n + \nu \mid n \in \mathbb{N}$, for an odd $y = p_1^{\beta_1} p_2^{\beta_2} \dots p_i^{\beta_i}$

where the exponents β_i and $p_i^{\beta_i}$ are as defined above and ν is a constant which depends on the value of y .

Remark 2. The variable $\mathbf{n}$ in the Definition 0.11 of $\mathbf{j}$:

$$\mathbf{j} = \mathbf{y} (\mathbf{p}_1^{s_1+1} \mathbf{p}_2^{s_2+1} \dots \mathbf{p}_l^{s_l+1}) \mathbf{n} + \nu$$

denotes the n -th element of the set for which the same coefficient $\mathbf{y} (\mathbf{p}_1^{s_1+1} \dots)$ applies. The constant ν depends on the value of $\mathbf{y}$ and $\mathbf{w}$, it equals $\nu = 0 \mid \mathbf{w} = 0$ and $\nu = -1 \mid \mathbf{w} = 8$, these are the simplest cases. The constant ν is independent of both, the value of $\mathbf{n}$ as well as of $\mathbf{x}$. Therefore, providing that the values of $\mathbf{y}$ and $\mathbf{w}$ remain unchanged, the constant ν is valid for any arbitrary $\mathbf{n}, \mathbf{x} \in \mathbb{N}$. For small values of an odd square free $\mathbf{y}$, in the case $\mathbf{w} = 2$ the constant ν is equal to:

$$(\mathbf{y}^2 - 1) / 4 = (\mathbf{y} + 1)((\mathbf{y} + 1)/4) - ((\mathbf{y} + 1)/2)$$

For other instances of $\mathbf{w}$ in this particular case; let's denote $\nu_i \mid w = i$ then, $\nu_1 = \nu_2/2$ and $\nu_i = \nu_1 \times i$. Some research is necessary to establish the facts, this however was outside of the scope of the current research.

The above definitions are used to formulate $\mathbf{m}$:

Definition 0.12.

$$\mathbf{m} = t^2 \mathbf{j} + \frac{w}{8} (t-1)(t+1) \mid \text{for } \mathbf{j} \in \mathbb{N} \cup \{0\}$$

Remark 3. Subject to (depending on the value of $\mathbf{w}$), $\mathbf{j}$ and $\mathbf{w}$ are not simultaneously equal to zero in the Definition 0.12 of $\mathbf{m}$.

1. INTRODUCTION

The earliest written evidence of the Congruent Number Problem is found in the Arab manuscript Al-Kazin dated c. 972 A.D. It concerns right angled triangles with sides a , b and c . Although the result $c^2 = a^2 + b^2$ is known as the Pythagoras' Theorem, it certainly has been known much earlier in India c. 800 B.C. The problem of constructing a right angled triangle with integral sides however, has already been known in ancient Babylonia. A cuneiform clay tablet (known as Plimpton 322) dated c. 1900 B.C. contains a table of integer triples a , b and c s.t. $c^2 = a^2 + b^2$. Much later in 1225 Leonardo Pisano Fibonacci in his book "Liber Quadratorum" uses the term "congruum" to denote an arithmetic progression of three rational squares:

Definition 1.1 (Congruent number).

A number $\mathbf{n} \in \mathbb{N}$ is congruent if there exists a number $\mathbf{r} \in \mathbb{Q} \mid \mathbf{r} - \mathbf{n}, \mathbf{r}, \mathbf{r} + \mathbf{n}$ are all squares of rational numbers.

The congruent number may also be defined as:

Definition 1.2 (Congruent number alternative definition).

A number $\mathbf{n} \in \mathbb{N}$ is called a congruent number, if there exists a rational right triangle with area $\mathbf{n} \in \mathbb{N}$ and there are rational

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Q} \mid \mathbf{a}, \mathbf{b}, \mathbf{c} > 0, \mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2 \text{ and } \mathbf{n} = \frac{\mathbf{ab}}{2}$$

The Congruent Number Problem (CNP) asks for the precise definition of all congruent numbers. CNP is one of the greatest and oldest challenges in number theory. Euclid gave the sufficiency condition in the Elements - Book X, Propositions 28 and 29:

Theorem 1.3 (Euclid's sufficiency condition). *Positive integers $\{a, b, c\}$ satisfying the Diophantine equation:*

$$(1.1) \quad a^2 + b^2 = c^2$$

are called Pythagorean Triple. It is necessary and sufficient if they satisfy:

$$(1.2) \quad a = 2uv, b = u^2 - v^2, c = u^2 + v^2$$

for a relatively prime and of opposite parity $u, v \in \mathbb{Z} \mid u > v$. Furthermore, all primitive Pythagorean Triples are of this form.

For a proof refer to e.g. (Shanks, 1978) pp. 141. This paper presents a different view of the CNP. The Pythagorean triple $\{a, b, c\}$ may be classified according to the difference between its two longest sides (TCC). This concept came to my mind, when I was on the second year of studies at The Monash University, Melbourne, Australia. The idea may be stated as:

An arbitrary odd number greater than one can be written as $2n + 1 \mid n \in \mathbb{N}$. Now,

$$\begin{aligned} (2n + 1)^2 &= 4n^2 + 4n + 1 = 4n(n + 1) + 1 \\ &= (2n(n + 1)) + (2n(n + 1) + 1) \quad \forall n \in \mathbb{N} \end{aligned}$$

From:

$$\begin{cases} (2n + 1)^2 = & 4n^2 + 4n + 1 \\ (2n(n + 1))^2 = & 4n^4 + 8n^3 + 4n^2 \\ (2n(n + 1) + 1)^2 = & (4n^4 + 8n^3 + 4n^2) + (4n^2 + 4n + 1) \end{cases}$$

Observe that:

$$(2n + 1)^2 + (2n(n + 1))^2 = (2n(n + 1) + 1)^2 \quad \forall n \in \mathbb{N}$$

Thus, PT is given by:

$$(1.3) \quad \{a, b, c\} = \begin{cases} a = & (2n + 1) \\ b = & (2n(n + 1)) \\ c = & (2n(n + 1) + 1) \end{cases}$$

As it may easily be seen, the triangles defined by the formulae 1.3, have the difference $c - b = 1$. This view of the CNP has some advantages. Thorough research of this issue permitted the discovery of the extension of the domain of the Euclid's parametrization of Pythagorean triples given by Theorem 1.3. The complete revision of the Euclid's sufficiency condition is presented in Theorem 3.4.

1.1. TCC and the Pythagorean triple primitives.

A Pythagorean triple $\{a, b, c\}$ with $GCD(a, b, c) = 1$ is called primitive, the sequence of the triples begins with:

$$\{3, 4, 5\}, \{5, 12, 13\}, \{7, 24, 25\}, \{8, 15, 17\}, \{9, 40, 41\}, \{11, 60, 61\}, \dots$$

The sequence of the TCC classes of the Pythagorean triangles begins:

$$1, 2, 8, 9, 18, 25, 32, 49, 50, 72, 81, 98, 121, 128, 162, 169, 200, 225, 242, 288, 289, \dots$$

This sequence can be sub-divided into two sub-sequences based on the parity.

(1) The even TCC sub-sequence is given by:

$$2, 8, 18, 32, 50, 72, 98, 128, 162, 200, 242, \dots = 2 \times 1^2, 2 \times 2^2, 2 \times 3^2, 2 \times 4^2, 2 \times 5^2, \dots$$

Let s denote the sequential index of a particular entry within the sub-sequence, then the even TCC sub-sequence may be represented by the sum:

$$(1.4) \quad \sum_{m=0}^{(s-1)} (4m+2) = 2(s-1)(s+1) + 2 = 2s^2$$

The corresponding formulae for the triangle sides a, b and c are:

$$(1.5) \quad \{a, b, c\} = \begin{cases} a = 6s^2 - 2s & \forall s \in \mathbb{N} \\ b = 8s^2 - 6s + 1 \\ c = 10s^2 - 6s + 1 \end{cases}$$

Which give the area $\mathcal{A}$ of the triangle:

$$\mathcal{A} = (2(s-1) + 1)(3(s-1)s + 2s)(4(s-1) + 3)$$

(2) The odd sub-sequence TCC is given by:

$$1, 9, 25, 49, 81, 121, 169, 225, 289, \dots = 1^2, 3^2, 5^2, 7^2, 9^2, 11^2, 13^2, 15^2, 17^2, \dots$$

Let s denote the sequential index of a particular entry within the sub-sequence, then the odd TCC sub-sequence is represented by the sum:

$$(1.6) \quad 1 + \sum_{m=0}^{(s-1)} 8m = 4(s-1)s + 1 = 4\left(s - \frac{1}{2}\right)^2 \quad \forall s \in \mathbb{N}$$

The corresponding formulae for the triangle sides a, b and c are:

$$(1.7) \quad \{a, b, c\} = \begin{cases} a = 2s^3 - 7s^2 + 29s - 13 & \forall s \in \mathbb{N} \\ b = \frac{1}{2}(s^4 - 6s^3 + 31s^2 - 74s + 168) \\ c = \frac{1}{2}(s^4 - 6s^3 + 39s^2 - 82s + 170) \end{cases}$$

Which give the area $\mathcal{A}$ of the triangle:

$$\mathcal{A} = \frac{1}{4}((2s-1)(s^2 - s + 12)(s^2 - 3s + 13)(s^2 - 5s + 14))$$

The primitive Pythagorean triple formulae based on the even/odd sub-sequence TCC i.e. equations 1.5 and 1.7, produce however only a single triangle from each particular TCC class. To obtain all possible triangles within a particular TCC class we need to implement the ESS/OSS formulae.

2. INTEGRAL PYTHAGOREAN TRIPLES

Considering the Pythagorean triangles with both integral sides and area, it may seem unnecessary to create two different classes of triangles where the criterion is the parity of TCC. However, systematic generation of rational triangles is then left to chance. The OSS formulae facilitate structured method of generation of the rational Pythagorean triangles. To enable seamless integration of the theory, the Euclid's sufficiency condition has to be revised to extend the domain for u and v . This task is done in two steps as stated in Theorem 2.4 (integral Pythagorean triples) and Theorem 3.4 (rational Pythagorean triples).

Let's conceptualize the integral Pythagorean triples as being arranged in an ordered *ESS/OSS* two-dimensional array. Then, the variable $\mathbf{s}$ denotes the rows, while the variable $\mathbf{k}$ denotes the columns of the array. Hence, in a given row there are Pythagorean triples, all of which define a triangle of precisely the same difference between the triangle sides $\mathbf{c} - \mathbf{b}$.

2.1. The ESS class formulae.

The even sub-sequence Pythagorean triples are generated by the class formulae:

Theorem 2.1 (Pythagorean triples with the TCC criterion even).

The Pythagorean triples $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, generating right triangles corresponding to the even sub-sequence progression formulae for all $\mathbf{k}, \mathbf{s} \in \mathbb{N} \mid \mathbf{k} \geq (\mathbf{s} + 1)$ are given by:

$$(2.1) \quad \begin{aligned} \mathbf{a} &= 2\mathbf{k}\mathbf{s} \\ \mathbf{b} &= \mathbf{k}^2 - \mathbf{s}^2 \\ \mathbf{c} &= \mathbf{k}^2 + \mathbf{s}^2 \end{aligned}$$

which produce a triangle with $\mathbf{d} = \mathbf{c} - \mathbf{b} = 2\mathbf{s}^2$ and the area:

$$\mathcal{A} = \mathbf{k}\mathbf{s} (\mathbf{k}^2 - \mathbf{s}^2)$$

Proof.

The expressions for the Pythagorean triples $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, given by 2.1 are precisely equal to the parametric equations of the Pythagorean triples given by Theorem 1.3. Consequently, the same proof applies. $\square$

The equations 2.1 generate both primitive and non-primitive triangles. The primitive Pythagorean triples are given by the following theorem:

Theorem 2.2 (Primitive Pythagorean triples with the TCC criterion even).

The formulae 2.1 given by Theorem 2.1 generate primitive Pythagorean triples $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, for a right triangle corresponding to the even sub-sequence progression formulae for all $\mathbf{k}, \mathbf{s} \in \mathbb{N} \mid \mathbf{k} \geq (\mathbf{s} + 1)$ subject to $\mathbf{k}$ and $\mathbf{s}$ being of opposite parity and $\mathbf{k}$ being relatively prime to $\mathbf{s}$, this means $\text{GCD}(\mathbf{k}, \mathbf{s}) = 1$. In all other cases the formulae 2.1 given by Theorem 2.1 generate non-primitive Pythagorean triples $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

Proof.

This fact follows from the Theorem 1.3. $\square$

2.2. The OSS class formulae.

The OSS sub-sequence formulae allow us to demonstrate the incompleteness of the domain specification of the Euclid's sufficiency condition.

Theorem 2.3 (Pythagorean triples with the TCC criterion odd).

The Pythagorean triples $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, that generate right triangles corresponding to the odd sub-sequence progression formulae for all $\mathbf{k}, \mathbf{s} \in \mathbb{N} \mid \mathbf{k} \geq \mathbf{s}$, with the difference $\mathbf{d} = 4 \left(\mathbf{s} - \frac{1}{2}\right)^2$, are given by:

$$(2.2) \quad \begin{aligned} \mathbf{a} &= 2(2\mathbf{k} + 1) \left(\mathbf{s} - \frac{1}{2}\right) \\ \mathbf{b} &= 2\mathbf{k}(\mathbf{k} + 1) - \left(2 \left(\mathbf{s} - \frac{1}{2}\right)^2 - \frac{1}{2}\right) \\ \mathbf{c} &= 2\mathbf{k}(\mathbf{k} + 1) + \left(2 \left(\mathbf{s} - \frac{1}{2}\right)^2 + \frac{1}{2}\right) \end{aligned}$$

which produce a triangle with $\mathbf{d} = \mathbf{c} - \mathbf{b} = 4\left(\mathbf{s} - \frac{1}{2}\right)^2$ and the area:

$$\mathcal{A} = (2\mathbf{k} + 1)(\mathbf{k} - \mathbf{s} + 1)(\mathbf{k} + \mathbf{s})(2\mathbf{s} - 1)$$

Proof.

Positive integers $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ satisfying the Diophantine equation:

$$(2.3) \quad \mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2$$

are called Pythagorean Triple. From Equations 2.2, the set of the triples $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ for all $\mathbf{k}, \mathbf{s} \in \mathbb{N} \mid \mathbf{k} \geq \mathbf{s}$ is given by:

$$\begin{aligned} \mathbf{a} &= 2(2\mathbf{k} + 1)\left(\mathbf{s} - \frac{1}{2}\right) = (2\mathbf{k} + 1)(2\mathbf{s} - 1) \\ \mathbf{b} &= 2\mathbf{k}(\mathbf{k} + 1) - \left(2\left(\mathbf{s} - \frac{1}{2}\right)^2 - \frac{1}{2}\right) = 2(\mathbf{k} - \mathbf{s} + 1)(\mathbf{k} + \mathbf{s}) \\ \mathbf{c} &= 2\mathbf{k}(\mathbf{k} + 1) + \left(2\left(\mathbf{s} - \frac{1}{2}\right)^2 + \frac{1}{2}\right) = 2\mathbf{k}(\mathbf{k} + 1) + 2\mathbf{s}(\mathbf{s} - 1) + 1 \end{aligned}$$

Clearly, for any $\mathbf{k}$ and $\mathbf{s}$ as defined above, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all integer valued. Furthermore,

$$(2.4) \quad \mathbf{a}^2 + \mathbf{b}^2 = (2\mathbf{k}(\mathbf{k} + 1) + 2\mathbf{s}(\mathbf{s} - 1) + 1)^2 = \mathbf{c}^2$$

and the area of the triangle given by:

$$(2.5) \quad \mathcal{A} = \frac{\mathbf{ab}}{2} = (2\mathbf{k} + 1)(\mathbf{k} - \mathbf{s} + 1)(\mathbf{k} + \mathbf{s})(2\mathbf{s} - 1)$$

is clearly an integer. Since the triples generated by the formulae satisfy the conditions, consequently they are classified as Pythagorean Triples generating right triangles with integral sides and area. This completes the proof. $\square$

Equations 2.2 generate both primitive as well as non-primitive Pythagorean triples.

2.3. Revision of the Euclid's sufficiency condition.

The extension of the domain of the Euclid's sufficiency condition is designed to seamlessly fit as an alternative of Theorem 2.3, anticipating its application in generation of the rational Pythagorean triangles.

Theorem 2.4 (Revised Euclid's sufficiency condition).

The sufficiency condition of Euclid 1.3, states that a triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ to be classified as a Pythagorean Triple must satisfy:

$$\mathbf{a} = 2\mathbf{uv}, \mathbf{b} = \mathbf{u}^2 - \mathbf{v}^2, \mathbf{c} = \mathbf{u}^2 + \mathbf{v}^2$$

for some relatively prime and of opposite parity $\mathbf{u}, \mathbf{v} \in \mathbb{Z} \mid \mathbf{u} > \mathbf{v}$. This condition needs to be extended to include certain rational multiples as given by the Definition 0.2:

$$\mathbf{u} = \left(\mathbf{k} + \frac{1}{2}\right)\sqrt{2}, \text{ and } \mathbf{v} = \left(\mathbf{s} - \frac{1}{2}\right)\sqrt{2}, \quad \text{for any } \mathbf{k}, \mathbf{s} \in \mathbb{N} \mid \mathbf{k} \geq \mathbf{s}$$

Proof.

From the sufficiency condition of Euclid 1.3 upon substitution for $\{u, v\}$ from the Definition 0.2 we have:

$$(2.6) \quad \begin{aligned} 2uv &= (2k+1)(2s-1) &= a \\ u^2 - v^2 &= 2(k-s+1)(k+s) &= b \\ u^2 + v^2 &= 2k(k+1) + 2s(s-1) + 1 &= c \\ uv(u^2 - v^2) &= (2k+1)(k-s+1)(k+s)(2s-1) &= \mathcal{A} \end{aligned}$$

and by Theorem 2.3, the triple $\{a, b, c\}$ satisfies the conditions to be classified as Pythagorean triple generating right triangles with integral sides and area. This completes the proof. $\square$

Similarly as in the case of Theorem 2.3, equations 2.6, generate both primitive and non-primitive Pythagorean Triples. The primitive Pythagorean triples are given by the following theorem:

Theorem 2.5 (OSS sub-sequence Primitive Pythagorean triples).

The formulae 2.2 given by Theorem 2.3 generate primitive Pythagorean triples $\{a, b, c\}$ corresponding to the OSS sub-sequence for all $k, s \in \mathbb{N} \mid k \geq s$ subject to k being relatively prime to $(2c_1 + 1)s - (c_1 + 1)$, for any $c_1 \in \mathbb{N} \cup \{0\}$, this means, subject to $GCD(k, (2c_1 + 1)s - (c_1 + 1)) = 1$. In all other cases the formulae generate non-primitive Pythagorean triples.

Proof.

The variables $\{u, v\}$ as given by the Definition 0.2 generate Pythagorean Triples in accordance with Theorem 2.4. Suppose that $k = (2c_1 + 1)s - (c_1 + 1)$. Substituting this value of k into u as given by the Definition 0.2 and computing:

$$\begin{aligned} a &= 2uv \\ b &= u^2 - v^2 \\ c &= u^2 + v^2 \end{aligned}$$

we obtain:

$$\begin{aligned} a &= (2c_1 + 1)(1 - 2s)^2 \\ b &= 2c_1(c_1 + 1)(1 - 2s)^2 \\ c &= (2c_1(c_1 + 1) + 1)(1 - 2s)^2 \end{aligned}$$

Clearly, since the $GCD(a, b, c)$ has now a common factor, consequently the generated Pythagorean Triples in such a case are not primitive. $\square$

Theorem 2.6 (Positive triangle area).

The triangle area produced by the formulae 2.2 given by Theorem 2.3 is positive for all $k, s \in \mathbb{N} \mid k \geq s$. The triangle area equals zero for all $k, s \in \mathbb{N} \mid k = s - 1$. For all $k, s \in \mathbb{N} \mid k \leq s - 2$ the resultant triangle area is negative.

Proof.

The area of the triangle produced by the formulae 2.2 equals:

$$\mathcal{A} = (2k+1)(k-s+1)(k+s)(2s-1)$$

Let $k = (s - m)$ for any $m \in \mathbb{N}$. Then the area equals:

$$\mathcal{A} = (1 - m)(2(s - m) + 1)(2s - m)(2s - 1)$$

which clearly equals zero for $m = 1$ and is negative for all other $m \in \mathbb{N}$. $\square$

On the basis of the formulae 2.2 given by Theorem 2.3 we may generate an infinite sequence of right triangles with a given square free area - copies of the first triangle (after computing the square free part).

Theorem 2.7 (Infinite sequence of triangles of some given square free area).

Implementing notation as defined in the section to denote the initial triangle (the first of some given area and $GCD(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 1$) within the TCC class. The infinite sequence of right triangles (copies of the first triangle located at $\{\mathbf{s}_I, \mathbf{k}_I\}$) with identical square free part of their area, is given by computing the values of $\mathbf{s}$ and $\mathbf{k}$ from the formula:

$$(2.7) \quad \mathbf{s}_T = (\mathbf{I}_T - 1)(2\mathbf{s}_I - 1) + \mathbf{s}_I$$

$$(2.8) \quad \mathbf{k}_T = (\mathbf{I}_T - 1)(2\mathbf{k}_I + 1) + \mathbf{k}_I$$

then implementing the formulae 2.2 given by Theorem 2.3.

Proof.

The area of the first triangle in the sequence generated by the formulae 2.2 equals:

$$(2.9) \quad \mathcal{A} = (2\mathbf{k} + 1)(\mathbf{k} - \mathbf{s} + 1)(\mathbf{k} + \mathbf{s})(2\mathbf{s} - 1)$$

Substituting $\mathbf{s}_T$ and $\mathbf{k}_T$ from the Definitions 2.7 and 2.8 into the equation 2.9 we obtain:

$$(2.10) \quad \begin{aligned} \mathcal{A}_T &= (2\mathbf{k}_T + 1)(\mathbf{k}_T - \mathbf{s}_T + 1)(\mathbf{k}_T + \mathbf{s}_T)(2\mathbf{s}_T - 1) \\ &= (1 - 2\mathbf{I}_T)^4 (2\mathbf{k}_I + 1)(\mathbf{k}_I - \mathbf{s}_I + 1)(\mathbf{k}_I + \mathbf{s}_I)(2\mathbf{s}_I - 1) = (1 - 2\mathbf{I}_T)^4 \mathcal{A}. \end{aligned}$$

This completes the proof. $\square$

3. RATIONAL PYTHAGOREAN TRIPLES

While the formulae given by Theorems 2.1, 2.2 and 2.4 easily generate all right triangles with integral sides and area; generating rational right triangles with with integral area systematically is a problem which acquires a completely different level. Rational sides of the triangle may be obtained in a number of ways. The easiest method is by a simple change of the variable $\mathbf{k}$, this method however is severely limited in its scope. More advanced methods, jointly capable to generate systematically all rational Pythagorean Triples are presented afterwards.

3.1. Rational Pythagorean triples: Change of the variable $\mathbf{k}$.

If we think of the rational Pythagorean triples as being arranged in an ordered two-dimensional array, then, similarly as in the case of integral triples the variable $\mathbf{s}$ denotes the rows, while the variable $\mathbf{k}$ denotes the columns of the array. Consequently, in any given row there are Pythagorean triples, all of which define a triangle of precisely the same difference between the triangle sides $\mathbf{c} - \mathbf{b}$. In this case the difference $\mathbf{c} - \mathbf{b}$ is integral and is precisely equal to the OSS case. Rational triangles may be generated by implementing the change of variable $\mathbf{k}$ of the form:

$$\mathbf{k} = 2\mathbf{m} + \frac{3}{2} \mid \mathbf{m} \in \mathbb{N} \cup \{0\}$$

every triangle produced by the OSS class formulae 2.2 implementing $\mathbf{k}$ of this form is a rational triangle with integral area. This is the easiest modification to implement to produce rational triangles.

Theorem 3.1 (OSS rational triangles).

Let $\mathbf{r} \in \mathbb{N} \mid \mathbf{r} \equiv 1 \pmod{2}$ be a multiplier used to scale the right triangle given by the formulae 2.2. Then, for $\mathbf{k} \in \mathbb{Q} \mid \mathbf{k} = 2\mathbf{m} + \frac{3}{2}$, $\mathbf{m} \in \mathbb{N} \cup \{0\}$, where $\mathbf{k} \geq (\mathbf{s} - \frac{1}{2})$ and $\mathbf{s} \in \mathbb{N}$, the set of the Pythagorean triples $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \mid \mathbf{c} - \mathbf{b} = \mathbf{r}(1 - 2\mathbf{s})^2$ is given by:

$$(3.1) \quad \begin{aligned} \mathbf{a} &= \mathbf{r} \left(2(2\mathbf{k} + 1) \left(\mathbf{s} - \frac{1}{2} \right) \right) \\ \mathbf{b} &= \mathbf{r} \left(2\mathbf{k}(\mathbf{k} + 1) - \left(2 \left(\mathbf{s} - \frac{1}{2} \right)^2 - \frac{1}{2} \right) \right) \\ \mathbf{c} &= \mathbf{r} \left(2\mathbf{k}(\mathbf{k} + 1) + \left(2 \left(\mathbf{s} - \frac{1}{2} \right)^2 + \frac{1}{2} \right) \right) \end{aligned}$$

which give the area $\mathcal{A}$ of the triangle:

$$\mathcal{A} = \mathbf{r}^2(2\mathbf{k} + 1)(\mathbf{k} - \mathbf{s} + 1)(\mathbf{k} + \mathbf{s})(2\mathbf{s} - 1)$$

The PT formulae produce rational sides for any suitable $\mathbf{r}$, however, it is obvious that for $\mathbf{r} > 1$ the PT produced are non-primitive since $\text{GCD}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq 1$.

Proof.

We consider the case $\mathbf{r} = 1$. Let $\mathbf{k} \in \mathbb{Q} \mid \mathbf{k} = 2\mathbf{m} + \frac{3}{2}$, $\mathbf{m} \in \mathbb{N} \cup \{0\}$, then from the formulae 3.1 we obtain:

$$(3.2) \quad \mathbf{a} = \mathbf{r} \left(2(2\mathbf{k} + 1) \left(\mathbf{s} - \frac{1}{2} \right) \right) = \mathbf{r}(4\mathbf{m} + 4)(2\mathbf{s} - 1)$$

Clearly for any $\mathbf{m}$ and any $\mathbf{s} \in \mathbb{N}$ side $\mathbf{a}$ of the triangle in 3.2 remains an even number. Next consider the adjacent side:

$$(3.3) \quad \begin{aligned} \mathbf{b} &= \mathbf{r} \left(2\mathbf{k}(\mathbf{k} + 1) - \left(2 \left(\mathbf{s} - \frac{1}{2} \right)^2 - \frac{1}{2} \right) \right) \\ &= \mathbf{r} \left(4 \left(\mathbf{m} + \frac{3}{4} \right) \left(2\mathbf{m} + \frac{5}{2} \right) - 2\mathbf{s}(\mathbf{s} - 1) \right) \end{aligned}$$

For any $\mathbf{m}$ and any $\mathbf{s} \in \mathbb{N}$ the first term of the product in 3.3 is clearly an odd integer, the second is a rational number while the third term is an even integer. Necessarily therefore, the adjacent side of the right triangle remains a rational number for any $\mathbf{m}$ and any $\mathbf{s} \in \mathbb{N}$. Finally, the hypotenuse:

$$(3.4) \quad \begin{aligned} \mathbf{c} &= \mathbf{r} \left(2\mathbf{k}(\mathbf{k} + 1) + \left(2 \left(\mathbf{s} - \frac{1}{2} \right)^2 + \frac{1}{2} \right) \right) \\ &= \mathbf{r} \left(4 \left(\mathbf{m} + \frac{3}{4} \right) \left(2\mathbf{m} + \frac{5}{2} \right) + (2\mathbf{s}(\mathbf{s} - 1) + 1) \right) \end{aligned}$$

For any $\mathbf{m}$ and any $\mathbf{s} \in \mathbb{N}$ the first term of the product in 3.4 is clearly an odd integer, the second is a rational number while the third term is an odd integer. Necessarily therefore, the hypotenuse remains a rational number for any $\mathbf{m}$ and any $\mathbf{s} \in \mathbb{N}$. This implies that in the case $\mathbf{r} = 1$ and $\mathbf{k} \in \mathbb{Q} \mid \mathbf{k} = 2\mathbf{m} + \frac{3}{2}$, $\mathbf{m} \in \mathbb{N} \cup \{0\}$ clearly $\mathbf{b}, \mathbf{c} \in \mathbb{Q}$, thus we obtain a rational triangle with the difference $\mathbf{c} - \mathbf{b} = (1 - 2\mathbf{s})^2$. In the case $\mathbf{r} \equiv 0 \pmod{2}$, the term $2\mathbf{m} + \frac{5}{2}$ in both 3.3 and

3.4 becomes an integer and so necessarily $\mathbf{b}, \mathbf{c} \in \mathbb{N}$. For any $\mathbf{r} \equiv 1 \pmod{2}$ we have $\mathbf{b}, \mathbf{c} \in \mathbb{Q}$. This completes the proof. $\square$

3.2. Rational Pythagorean triples: triangles with $\mathbf{c} - \mathbf{b} = \frac{1}{t} \mid t \in \mathbb{Q}$.

In this case the rows and columns of the two-dimensional array containing the rational Pythagorean triples change. The rows are denoted by $1/t$, while the columns are denoted by the variable $\mathbf{j}$ in the case $t \in \mathbb{N}$ or the variable $\mathbf{n}$ (refer to the Definition 0.6 of $\mathbf{m}$ and Definition 0.11 of $\mathbf{j}$ respectively) in the case $t \in \mathbb{Q}$. Similarly as in the preceding cases, in any given row there are Pythagorean triples, all of which define a triangle of precisely the same difference between the triangle sides $\mathbf{c} - \mathbf{b}$.

Theorem 3.2 (Rational right triangles Part 1).

Let $\mathbf{t} \in \mathbb{N} \mid \mathbf{t} > 1$ be a factor used to define the difference between the two longest sides of the right Pythagorean triangle given by the formulae 2.2. Then, implementing $\mathbf{q}, \mathbf{z}, \mathbf{w}$ and $\mathbf{m}$ as given by the Definitions 0.3, 0.4, 0.5, and 0.6 respectively we have:

$$\mathbf{m} = \mathbf{z}^2 \mathbf{j} + \frac{\mathbf{w}}{8} (\mathbf{z} - 1)(\mathbf{z} + 1) \mid \text{for } \mathbf{j} \in \mathbb{N} \cup \{0\}$$

Subject to (depending on the value of $\mathbf{w}$), $\mathbf{j}$ and $\mathbf{w}$ are not simultaneously equal to zero in the definition of $\mathbf{m}$. Next, implementing the formulae 2.2 (or alternatively Definition 0.2 and Euclid's parametric equations) and variables $\mathbf{k}$ and $\mathbf{s}$ as given by the Definitions 0.7 and 0.8:

$$\mathbf{k} = \frac{1}{2\mathbf{t}} \left(-\mathbf{t} + (8\mathbf{m} + \mathbf{q}) \sqrt{\mathbf{t}} \right) \mid, \text{ with } \mathbf{m} \text{ as given above.}$$

$$\mathbf{s} = \frac{1}{2} + \frac{1}{2\sqrt{\mathbf{t}}}$$

we may generate the set of the rational Pythagorean triples $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \mid \mathbf{c} - \mathbf{b} = \frac{1}{t}$.

Proof.

In the proof we implement Theorem 2.4. Substituting into the Definition 0.2 variables $\mathbf{k}$ and $\mathbf{s}$ from the Definitions 0.7 and 0.8 respectively, we obtain:

$$\mathbf{u} = \frac{(8\mathbf{j} + \mathbf{w}) \mathbf{z}^2 + (\mathbf{q} - \mathbf{w})}{\sqrt{2}\sqrt{\mathbf{t}}}, \text{ and } \mathbf{v} = \frac{1}{\sqrt{2}\sqrt{\mathbf{t}}}$$

First we verify that the above substitution produces right angled Pythagorean triangles. From the Euclid's sufficiency condition we have:

$$(3.5) \quad \mathbf{a} = 2\mathbf{uv}, \mathbf{b} = \mathbf{u}^2 - \mathbf{v}^2, \mathbf{c} = \mathbf{u}^2 + \mathbf{v}^2$$

Hence evaluating:

$$(3.6) \quad \mathbf{a} = \frac{(8\mathbf{j} + \mathbf{w}) \mathbf{z}^2 + (\mathbf{q} - \mathbf{w})}{t}$$

$$(3.7) \quad \mathbf{b} = \frac{((8\mathbf{j} + \mathbf{w}) \mathbf{z}^2 + (\mathbf{q} - \mathbf{w}))^2 - 1}{2t}$$

$$(3.8) \quad \mathbf{c} = \frac{((8\mathbf{j} + \mathbf{w}) \mathbf{z}^2 + (\mathbf{q} - \mathbf{w}))^2 + 1}{2t}$$

It is evident that for $q, t, z \in \mathbb{N}$ and $j, w \in \mathbb{N} \cup \{0\}$ the triangle sides are rational. Furthermore, from 3.7 and 3.8 we may compute the difference $c - b = 1/t$. Next, we verify:

$$a^2 + b^2 = \frac{((8j + w)z^2 + (q - w))^2 \left(((8j + w)z^2 + (q - w))^2 + 2 \right) + 1}{4t^2} = c^2$$

Clearly, the substitution produces right angled Pythagorean triangles. Now we need to determine if the triangle area $\mathcal{A} \in \mathbb{N}$. In the proof we only consider two cases from the definition 0.5. The first is the case $z = t$ and the second is the case $z = \frac{t}{2}$. All other cases from the definition 0.5 are left as an exercise for the reader. The area of the triangle is given by:

$$(3.9) \quad \mathcal{A} = uv(u^2 - v^2) \\ = \frac{((8j + w)z^2 + (q - w - 1))((8j + w)z^2 + (q - w))((8j + w)z^2 + (q - w + 1))}{4t^2}$$

In the first case, after substituting:

$$q = 3, w = q, t = 2r + 1 \text{ for some } r \in \mathbb{N}, z = t$$

into 3.9, we obtain:

$$(3.10) \quad \mathcal{A} = 2 \left(2j(2r + 1)^2 + 3r(r + 1) + 1 \right) \left(4j(2r + 1)^2 + 6r(r + 1) + 1 \right) (8j + 3)$$

which for $j, r \in \mathbb{N}$ evidently is an integer. The second case requires substituting:

$$q = 1, w = q - 1, t = 4r \text{ for some } r \in \mathbb{N}, z = \frac{t}{2}$$

into 3.9, upon which we obtain:

$$\mathcal{A} = (32jr^2 + 1)(16jr^2 + 1)j$$

which for $j, r \in \mathbb{N}$ clearly is an integer. Since the triples $\{a, b, c\}$ generated by the formulae satisfy the conditions, consequently they are classified as Pythagorean Triples which generate rational right triangles with an integral area. Due to arbitrariness of t , the formulae generate all possible Pythagorean triangles satisfying the condition $c - b = 1/t \mid t \in \mathbb{N}$, and $t > 1$. This completes the proof. $\square$

Next Theorem concerns a more general case of m i.e. $t \in \mathbb{Q}$.

Theorem 3.3 (Rational right triangles Part 2).

Let $t \in \mathbb{Q}$ (*in lowest terms*) be a factor used to define the difference between the two longest sides of the right Pythagorean triangle given by the formulae 2.2. Thus, for $x, y \in \mathbb{N}$ we implement t, q, w and j in accordance with the Definitions 0.9, 0.3, 0.10 and 0.11 respectively:

The above definitions are used to formulate m as given by the Definition 0.12:

$$m = t^2 j + \frac{w}{8} (t - 1)(t + 1) \mid \text{ for } j \in \mathbb{N} \cup \{0\}$$

Subject to (depending on the value of w), j and w are not simultaneously equal to zero in the Definition of m . Next, implementing the formulae 2.2 (or alternatively Definition 0.2) and variables k and s given by the Definitions: 0.7 and 0.8 of the form:

$$(3.11) \quad k = \frac{1}{2t} \left(-t + (8m + q) \sqrt{t} \right) \mid, \text{ with } m \text{ as given above.}$$

$$(3.12) \quad s = \frac{1}{2} + \frac{1}{2\sqrt{t}}$$

we may generate the set of the rational Pythagorean triples $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \mid \mathbf{c} - \mathbf{b} = \frac{y}{x}$.

Proof.

The proof implements Theorem 2.4. Substituting into the Definition 0.2, variables $\mathbf{k}$ and $\mathbf{s}$ as given in equations 3.11 and 3.12 respectively we obtain:

$$\mathbf{u} = \frac{(8j+w)x^2 + (q-w)y^2}{\sqrt{2}\sqrt{\frac{x}{y}}y^2}, \text{ and } \mathbf{v} = \frac{1}{\sqrt{2}\sqrt{\frac{x}{y}}}$$

First we verify if the above substitution for $\mathbf{u}$ and $\mathbf{v}$ produces right angled Pythagorean triangles. From the Euclid's sufficiency condition we have:

$$\mathbf{a} = 2\mathbf{uv}, \mathbf{b} = \mathbf{u}^2 - \mathbf{v}^2, \mathbf{c} = \mathbf{u}^2 + \mathbf{v}^2$$

Hence evaluating:

$$(3.13) \quad \mathbf{a} = \frac{(8j+w)x}{y} + \frac{(q-w)y}{x}$$

$$(3.14) \quad \mathbf{b} = \frac{(8j+w)^2 x^4 + 2(q-w)(8j+w)x^2 y^2 + ((q-w)^2 - 1)y^4}{2xy^3}$$

$$(3.15) \quad \mathbf{c} = \frac{(8j+w)^2 x^4 + 2(q-w)(8j+w)x^2 y^2 + ((q-w)^2 + 1)y^4}{2xy^3}$$

It is evident that for $q, x, y \in \mathbb{N}$ and $j, w \in \mathbb{N} \cup \{0\}$ the triangle sides are rational. Furthermore, from 3.14 and 3.15 we may compute the difference $\mathbf{c} - \mathbf{b} = 1/t = y/x$. Next, we verify:

$$\mathbf{a}^2 + \mathbf{b}^2 = \frac{\left((8j+w)^2 x^4 + 2(q-w)(8j+w)x^2 y^2 + ((q-w)^2 + 1)y^4\right)^2}{4x^2 y^6} = \mathbf{c}^2$$

Clearly, the substitution produces right angled Pythagorean triangles. Next we need to determine if the triangle area $\mathcal{A} \in \mathbb{N}$. In the proof we only consider one case from the definition 0.10, i.e. the case $x \equiv 0 \pmod{2}$. All other cases from the Definition 0.10 are left as an exercise for the reader. The area of the triangle is given by:

$$(3.16) \quad \mathcal{A} = uv(u^2 - v^2) = \frac{((8j+w)x^2 + (q-w-1)y^2)((8j+w)x^2 + (q-w)y^2)((8j+w)x^2 + (q-w+1)y^2)}{4x^2 y^4}$$

Now, substituting (please refer to the Definition 0.10):

$$q = 1, w = q - 1$$

into 3.16, we obtain:

$$(3.17) \quad \mathcal{A} = \frac{(4j(4jx^2 + y^2)(8jx^2 + y^2))}{y^4} = 4j + \frac{(128j^3 x^4)}{y^4} + \frac{(48j^2 x^2)}{y^2}$$

Since $q = 1$ it implies that $w = 0$ in which case $\nu = 0$ in the Definition 0.11 of j . Since y is odd in this case (please refer to the Definition 0.10), case 3 of the Definition 0.11 applies. Clearly, substituting:

$$(3.18) \quad j = y(p_1^{s_1+1} p_2^{s_2+1} \dots p_i^{s_i+1})$$

into the equation 3.17 shows that it is an integer, for any $x, y \in \mathbb{N}$. Since the triples $\{a, b, c\}$ generated by the formulae satisfy the conditions, consequently they are classified as Pythagorean Triples which generate rational right triangles with an integral area. Due to arbitrariness of t , the formulae generate all possible Pythagorean triangles satisfying the condition $c - b = y/x \mid x, y \in \mathbb{N}$, and $x, y > 1$. This completes the proof. $\square$

3.3. Complete Euclid's sufficiency condition.

The extension of the domain for the Euclid's sufficiency condition involving both u and v as given by Theorem 2.4, produces Pythagorean triples which generate right triangles with both integral sides and the area. To generate rational Pythagorean triples in a systematic way, further modification is necessary. We have to extend the domain of $k, s \in \mathbb{N}$ to include certain values of $k, s \in \mathbb{R}$.

Theorem 3.4 (Euclid's sufficiency condition).

The sufficiency condition of Euclid 1.3, states that a triple $\{a, b, c\}$ to be classified as a Pythagorean Triple must satisfy:

$$a = 2uv, b = u^2 - v^2, c = u^2 + v^2$$

for some relatively prime and of opposite parity $u, v \in \mathbb{Z} \mid u > v$. This condition has been extended in Theorem 2.4 to include certain rational multiples:

$$(3.19) \quad u = \left(k + \frac{1}{2}\right) \sqrt{2}, \text{ and } v = \left(s - \frac{1}{2}\right) \sqrt{2}, \quad \text{for any } k, s \in \mathbb{N} \mid k \geq s$$

The extension of the Definition of both u as well as v to include the rational Pythagorean triangles involves substitution for the variables k and s as given by the Definitions 0.7 and 0.8 (m given by the Definitions 0.6 or 0.12), of the form:

$$k = \frac{1}{2t} \left(-t + (8m + q) \sqrt{t}\right), \text{ and } s = \frac{1}{2} + \frac{1}{2\sqrt{t}}$$

Proof. The proof follows from Theorems 3.2 and 3.3 $\square$

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