A REDUCED BASIS TECHNIQUE FOR LONG-TIME UNSTEADY TURBULENT FLOWS

optimal parameter sampling and constrained projection

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OBJECTIVE

Develop a reduced basis technique for long-time integration of parameterized incompressible turbulent flows.

MODEL PROBLEM

We consider the **lid-driven cavity problem**:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \frac{1}{\text{Re}} \Delta u + \nabla p = 0 & \Omega \times \mathbb{R}_+ \\ \nabla \cdot u = 0 & \Omega \times \mathbb{R}_+ \\ u|_{\partial\Omega} = g, \quad u(0) = 0 \end{cases}$$

$$g = (1 - x_1^2)^2 \mathbf{e}_1 \quad \Omega = (-1, 1)^2$$

$\text{Re} \in \mathcal{P} = [15000, 25000]$

The flow exhibits a long-time unsteady behavior for all values of the Reynolds number in the parameter range.

OFFLINE/ONLINE PARADIGM

We seek an estimate of $u(\text{Re})$ of the form

$$\hat{u}(x, t; \text{Re}) = R_g + \sum_{n=1}^N a_n(t; \text{Re}) \zeta_n(x) \quad \forall \text{Re} \in \mathcal{P}$$

where $R_g \in [H^1(\Omega)]^2$, $\nabla \cdot R_g \equiv 0$ is a suitable lift.

Offline stage: given $\{u(\cdot; t^k; \text{Re}_\ell)\}_{k,\ell}$, $\{t^k\}_{k=1}^K \subset (0, T)$, $\{\text{Re}_\ell\}_{\ell=1}^L \subset \mathcal{P}$,

- generate the reduced space $\mathcal{Z}^u = \{\zeta_n\}_{n=1}^N$;
- formulate the Reduced Order Model.

Online stage: given $\text{Re} \in \mathcal{P}$, estimate the coefficients $\mathbf{a}(t) = [a_1(t), \dots, a_N(t)]$, and use them to compute QOIs:

mean flow $\langle u \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T u(t) dt$

TKE $\text{TKE}(t) = \int_\Omega \|u(t) - \langle u \rangle\|_2^2 dt$

CHALLENGES

Turbulent flows present several challenges including

- inherently high-dimensional dynamics;
- chaotic behavior in time;
- several spurious behaviors for truncated approximations;
- no reliable *a posteriori* error estimators.

GALERKIN PROJECTION VS CONSTRAINED PROJECTION

Given the time grid $\{t^j = j\Delta t\}_{j=0}^J$.

Galerkin ROM: (Galerkin) find $\mathbf{a}^{j+1}$ such that

$$\mathbb{A}(\mathbf{a}^j) \mathbf{a}^{j+1} = \mathbf{F}(\mathbf{a}^j)$$

constrained Galerkin ROM: (cGalerkin) find $\mathbf{a}^{j+1}$ to minimize

$$\min_{\mathbf{a}} \|\mathbb{A}(\mathbf{a}^j) \mathbf{a} - \mathbf{F}(\mathbf{a}^j)\|_2^2$$

subjected to $\alpha_n \leq a_n \leq \beta_n$

Constraints are used to incorporate prior information about the long-time attractor, and are determined using the snapshots ($\Rightarrow$ *sample minima* and *sample maxima*).

Remark 1: if $\mathbf{a}_{\text{Gal}}^{j+1} = \mathbb{A}(\mathbf{a}^j)^{-1} \mathbf{F}(\mathbf{a}^j)$ satisfies the constraints, then Galerkin=cGalerkin.

Remark 2: for semi-implicit and explicit time discretizations, cGalerkin corresponds to a *convex quadratic programming* problem, which can be solved using interior point methods.

TIME-AVERAGED A POSTERIORI ERROR INDICATOR

Given $\{w^j\}_j \subset V_{\text{div}}$, we define the residual $e : V_{\text{div}} \times V_{\text{div}} \times V_{\text{div}} \rightarrow \mathbb{R}$ associated with the discretized Navier-Stokes equations at time t^j

$$e(w^j, w^{j+1}, v; \text{Re}) = \int_\Omega \left(\frac{w^{j+1} - w^j}{\Delta t} + (w^j + R_g) \cdot \nabla (w^{j+1} + R_g) \right) \cdot v + \frac{1}{\text{Re}} \nabla (w^{j+1} + R_g) : \nabla v dx.$$

Then, we define the **time-averaged residual** $\langle R \rangle \left(\{w^j\}_{j=0}^J, v; \text{Re} \right) = \frac{\Delta t}{T - T_0} \sum_{j=J_0}^{J-1} e(w^j, w^{j+1}, v; \text{Re})$,

where $T = t^J$ and $T_0 = t^{J_0}$, and finally we introduce the **time-averaged *a posteriori* error indicator**:

$$\Delta^u \left(\{w^j\}_{j=0}^J; \text{Re} \right) := \|\langle R \rangle \left(\{w^j\}_{j=0}^J, \cdot; \text{Re} \right)\|_{V'_{\text{div}}}.$$

Remark 1: if $J - J_0 \rightarrow \infty$, $\langle R \rangle$ converges to the RANS residual. For this reason, we expect Δ^u to be highly correlated with the error in mean flow prediction.

Remark 2: Δ^u admits an offline/online computational decomposition ongoing work to reduce memory costs

Remark 3: we do not expect Δ^u to be in good quantitative agreement with $\|\langle u \rangle - \langle \hat{u} \rangle\|_{V_{\text{div}}}$. Corrections are available to improve effectivity.

POD-*h*GREEDY SAMPLING/APPROXIMATION (EFTANG ET AL., 2011)

We consider the following procedure for the construction and the evaluation of the ROM.

Offline:

Pick $\text{Re}_1^* \in \mathcal{P}$ randomly;

For $\ell = 1, \dots, L$

run the DNS solver for $\text{Re} = \text{Re}_\ell^*$, and then apply POD based on the DNS snapshots to build the ℓ -th ROM;

run the ROM for all $\text{Re} \in \mathcal{P}_{\text{train}}$ and compute $\Delta^u(\text{Re})$;

select $\text{Re}_{\ell+1}^* = \arg \max_{\text{Re} \in \mathcal{P}_{\text{train}}} \min_{\ell'=1, \dots, \ell} \Delta_{\ell'}^u(\text{Re})$;

EndFor

Online: given $\text{Re} \in \mathcal{P}$,

run $n_{\text{cand}} \leq L$ ROMs built offline and compute the error indicator;

return the solution that minimizes the error indicator.

Remark 1: *h*-refinement is motivated by the difficulty in combining modes from different parameters.

RESULTS: $\text{Re} = 15000$

Solution reproduction problem: given DNS data for $\text{Re} = 15000$, we run our ROM to reproduce the DNS statistics.

RESULTS: $\text{Re} \in [15000, 25000]$

Parametric problem: we execute three steps of the POD-*h*Greedy algorithm starting from $\text{Re}_1^* = 15000$. We consider $\mathcal{P}_{\text{train}} = \{(15 + k) \cdot 10^3\}_{k=0}^{10}$.

The error indicator is a good indicator: for 10/11Re the same ROM minimizes both the relative error and Δ^u .

The error is below 13% for all values of the Reynolds number considered.

REFERENCES

- [1] L Fick, Y Maday, AT Patera, T Taddei "A stabilized POD model for turbulent flows over a range of Reynolds numbers: optimal parameter sampling and constrained projection", *JCP*(submitted), 2017.