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THE COMPLEXIFICATION OF THE EXCEPTIONAL JORDAN ALGEBRA AND APPLICATIONS TO PARTICLE PHYSICS

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Abstract

Recent papers of Todorov and Dubois-Violette[4] and Krasnov[7] contributed revitalizing the study of the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ in its relations with the true Standard Model gauge group G_{SM} . The absence of complex representations of F_4 does not allow $\text{Aut}(\mathfrak{h}_3(\mathbb{O}))$ to be a candidate for any Grand Unified Theory, but the group of automorphisms of the complexification of this algebra, i.e. $\mathfrak{h}_3^{\mathbb{C}}(\mathbb{O})$, is isomorphic to the compact form of E_6 . Following Boyle in [12], it is then easy to show that the gauge group of the minimal left-right symmetric extension of the Standard Model is isomorphic to a proper subgroup of $\text{Aut}(\mathfrak{h}_3^{\mathbb{C}}(\mathbb{O}))$.

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1 INTRODUCTION

Recent papers of Todorov and Dubois-Violette[4] and Krasnov[7] characterized the Standard Model gauge group G_{SM} as a subgroup of automorphisms of the exceptional Jordan algebra $\mathfrak{h}_3(0)$. These works, along with previous of Baez and Huerta[3], revitalized attention on the role octonions might have in the characterization of the Standard Model true gauge group G_{SM} , i.e. $[SU(3) \times SU(2) \times U(1)]/\mathbb{Z}_6$. One of the main issues with this approach to the Standard Model is that all groups related with Octonions, such as G_2 and F_4 , do not have complex representation. Recently, Boyle[12] proposed the analysis of the complexification of the Albert algebra, i.e. $\mathfrak{h}_3^{\mathbb{C}}(0)$, whose automorphisms are isomorphic to the compact form of E_6 and that evidenced a relation with the gauge group of the minimal left-right symmetric extension of the Standard Model G_{LR} , i.e. $[SU(3) \times SU(2)_L \times SU(2)_R \times U(1)]/\mathbb{Z}_6$, along with the E_6 and Spin(10) unification.

In section 2 we introduce the normed division algebras through the Cayley-Dickson construction. We then focus on the algebra of Octonions, their automorphisms and a generalization of complex analysis on Octonions developed by Gentili and Struppa in [8]. In section 3 we introduce Jordan algebras and focus on the exceptional Jordan algebra $\mathfrak{h}_3(0)$, while in section 4 we overview some of the possible applications to particle physics. Specifically we show the canonical isomorphism between $\mathfrak{h}_2(\mathbb{K})$ and the Minkowski space-time for every normed division algebra $\mathbb{K}$, we then proceed showing the relevance of the $\mathfrak{h}_3(0)$ algebra and its complexification for the gauge group of the Standard Model and its minimal left-right symmetric extension.

2 THE OCTONIONS

Octonions $\mathbb{O}$ are, along with Real numbers $\mathbb{R}$, Complex numbers $\mathbb{C}$ and Quaternions $\mathbb{H}$, one of the four *normed division algebras*. In this section we will define these algebras through the Cayley-Dickson construction and present some fundamental facts on Octonions, their geometry and their analysis.

2.1 NORMED DIVISION ALGEBRAS

An *algebra* is a real vector space A with a bilinear multiplication. We will assume our algebras to be unital, i.e. it exists an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$. If

$$(ab)c = a(bc), \tag{1}$$

for every $a, b, c \in A$ the algebra is called *associative*, and if $ab = ba$ for every $a, b \in A$ then the algebra is called *commutative*. Finally, if the subalgebra generated by any two elements of A is associative then the algebra is called *alternative*.

Our focus on this section it will be on normed division algebras. A *normed division algebra* is a unital algebra with a real valued function $|\cdot|$ called *norm* from A to $[0, \infty)$ such that

$$|ab| = |a| |b|. \quad (2)$$

In particular, a normed division algebra is a *division algebra*, since (2) implies $ab = 0$ if and only if $a = 0$ or $b = 0$.

The Cayley-Dickson construction. Normed division algebras are better understood from the perspective of a more general kind of algebra, namely $\ast$ -*algebra*, that are algebras equipped with a *conjugation* $\ast$, i.e. a $\mathbb{R}$ -linear map such that

$$a^{\ast\ast} = a, \quad (ab)^{\ast} = b^{\ast} a^{\ast}, \quad (3)$$

for all $a, b \in A$. We say that the algebra is *real* if $a^{\ast} = a$ for all of its elements and if we also have

$$a^{\ast} + a \in \mathbb{R}, \quad \text{and} \quad a^{\ast} a = a a^{\ast} > 0, \quad (4)$$

we say that the algebra is *nicely normed*. In this case we can define a norm on A by

$$|a|^2 = a a^{\ast}. \quad (5)$$

Moreover we can define the inverse a^{-1} as

$$a^{-1} = a^{\ast} / |a|^2. \quad (6)$$

We thus have (see [2]) that all nicely normed and alternative $\ast$ -*algebra* are normed division algebras.

Let A be a $\ast$ -algebra, then we can define a new $\ast$ -algebra A' with elements $(a, b) \in A \times A$, with multiplication given by

$$(a, b)(c, d) = (ac - db^{\ast}, a^{\ast}d + cb), \quad (7)$$

and with a new conjugation given by

$$(a, b)^{\ast} = (a^{\ast}, -b). \quad (8)$$

This procedure is called the *Cayley-Dickson construction* and it's easy to see that starting with the real algebra $\mathbb{R}$ it produces four normed division algebras, i.e.

$$\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O}. \quad (9)$$

The four normed division algebras. A well known theorem due to Hurwitz[1] states that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, i.e. the algebras that can be produced through the Cayley-Dickson construction, are the only four normed division algebras. More explicitly, since specific properties are lost every time we apply the Cayley-Dickson construction[2], we have the following

Figure 1: Multiplication rule of Octonions $\mathbb{O}$ as real vector space $\mathbb{R}^8$ in the basis $\{i_0 = 1, i_1, \dots, i_7\}$.

- $\mathbb{R}$ is a real commutative associative nicely normed;
- $\mathbb{C}$ is a commutative associative nicely normed;
- $\mathbb{H}$ is an associative nicely normed;
- $\mathbb{O}$ is an alternative nicely normed.

2.2 THE OCTONIONS

The Octonions are the only non-associative normed division algebra. A practical way to work with them is to consider their $\mathbb{R}^8$ decomposition. Let $x \in \mathbb{O}$ and let $\{i_0 = 1, i_1, \dots, i_7\}$ be a basis of $\mathbb{R}^8$, then we can encode the octonian multiplication through the following relations, i.e.

$$i_\alpha i_\beta = -\delta_{\alpha\beta} + \epsilon_{\alpha\beta\gamma} i_\gamma, \quad (10)$$

where $\delta_{\alpha\beta}$ is the Kroenecker delta and $\epsilon_{\alpha\beta\gamma}$ is totally antisymmetric in α, β, γ , non zero and equal to 1 in the following set

$$\sigma = \{(1, 2, 3), (1, 4, 5), (1, 7, 6), (2, 6, 4), (2, 7, 5), (3, 5, 6), (3, 7, 4)\}, \quad (11)$$

which can be mnemonically encoded through the *Fano plane* (Fig. 1).

We therefore have

$$x = x_0 + \sum_{k=1}^7 x_k i_k, \text{ and } x^* := \bar{x} = x_0 - \sum_{k=1}^7 x_k i_k, \quad (12)$$

and the expression for the norm squared

$$|x|^2 = x\bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2. \quad (13)$$

Following [8] we note that a suitable basis for $\mathbb{O}$ is

$$i_0 = 1, i_1, i_2, i_1 i_2, i_4, i_1 i_4, i_2 i_4, (i_1 i_2) i_4, \quad (14)$$

and therefore writing

$$x = (x_0 + x_1 i_1) + (x_2 + x_3 i_1) i_2 + [(x_4 + x_5 i_1) + (x_6 + x_7 i_1) i_2] i_4, \quad (15)$$

we have a decompositions of the Octonions given by

$$\begin{aligned} \mathbb{O} &\cong \mathbb{C} \oplus \mathbb{C} i_2 \oplus \mathbb{C} i_4 \oplus \mathbb{C} i_2 i_4 \\ &\cong \mathbb{H} \oplus \mathbb{H} i_4. \end{aligned} \quad (16)$$

Since the algebra of Octonions is alternative, any product involving two independent octonions do associate, i.e.

$$(xy)y = xy^2, \quad (17)$$

$$(xy)x = x(yx), \quad (18)$$

$$(xy)\bar{y} = x|y|^2, \quad (19)$$

$$(xy)\bar{x} = x(y\bar{x}). \quad (20)$$

In fact, any two independent octonions generate an algebra isomorphic to the quaternions $\mathbb{H}$. Another consequence of alternativity are the *Moufang identities*, i.e.

$$(xyx)z = x(y(xz)), \quad (21)$$

$$z(xyx) = ((xz)y)x, \quad (22)$$

$$(xy)(zx) = x(yz)x, \quad (23)$$

therefore, even though octonions under multiplications are not a group since they lack of associativity, nevertheless they form a *Moufang loop*. The same happens to $\mathbb{S} \subset \mathbb{O}$, the octonionic unit sphere, i.e.

$$\mathbb{S} = \left\{ x \in \mathbb{O} : |x|^2 = 1 \right\}, \quad (24)$$

which again is a Moufang loop under multiplication.

2.3 AUTOMORPHISMS OF THE OCTONIONS

Let A be an algebra, we define $\text{Aut}(A)$ the grup of automorphism of A , i.e. the group whose elements are the transformation $\varphi : A \longrightarrow A$ such that

$$\varphi(xy) = \varphi(x)\varphi(y), \quad (25)$$

and whose multiplication is given by composition of transformation. It is straightforward to see that automorphisms of the Quaternionic algebra $\mathbb{H}$ is given by the group of norm-preserving and orientation-preserving transformations on the vector space of imaginary elements of $\mathbb{H}$, i.e.

$$\text{Aut}(\mathbb{H}) \cong SO(3). \quad (26)$$

Similarly, we could argue that any automorphism of the Octonions $\mathbb{O}$ is given by a group of norm-preserving and orientation-preserving transformation on the vector space of imaginary elements of $\mathbb{H}$, i.e. $\text{Aut}(\mathbb{O}) \subseteq SO(7)$, but, since (14) the imaginary units i_1, i_2 and i_4 generates all the Octonionic algebra, the remaining imaginary units, i.e. i_3, i_5 and i_7 , are already constrained by the automorphism value on i_1, i_2 and i_4 .

Following [9], $\text{Aut}(\mathbb{O})$ is a Lie group of dimension 14 whose generators $\{A_{i_k}, G_{i_k}\}_{k=1,\dots,7}$ are rotations in two planes that point to the same imaginary unit i_k . The generators A_{i_k} are rotations that leave unchanged other two imaginary units, e.g.

$$A_{i_4} : \begin{aligned} i_1 &\rightarrow i_1 \cos \alpha - i_2 \sin \alpha, \\ i_2 &\rightarrow i_1 \sin \alpha + i_2 \cos \alpha, \\ i_3 &\rightarrow i_3, \\ i_4 &\rightarrow i_4, \\ i_5 &\rightarrow i_5 \cos \alpha + i_6 \sin \alpha, \\ i_6 &\rightarrow i_5 \sin \alpha - i_6 \cos \alpha, \\ i_7 &\rightarrow i_7, \end{aligned} \quad (27)$$

while G_{i_k} leave unchanged only i_k , e. g.

$$G_{i_4} : \begin{aligned} i_1 &\rightarrow i_1 \cos \alpha - i_2 \sin \alpha, \\ i_2 &\rightarrow i_1 \sin \alpha + i_2 \cos \alpha, \\ i_3 &\rightarrow i_7 \cos 2\alpha - i_3 \sin 2\alpha, \\ i_4 &\rightarrow i_4, \\ i_5 &\rightarrow i_5 \cos \alpha - i_6 \sin \alpha, \\ i_6 &\rightarrow i_5 \sin \alpha + i_6 \cos \alpha, \\ i_7 &\rightarrow i_7 \sin 2\alpha + i_3 \cos 2\alpha, \end{aligned} \quad (28)$$

It is worth noting that $\text{Aut}(\mathbb{O})$ is isomorphic to the exceptional Lie group G_2 and that $\{A_{i_1}, \dots, A_{i_7}, G_{i_7}\}$ generate a subgroup in G_2 isomorphic to $SU(3)$ that will be used in the last section of this text.

2.4 OCTONIONIC ANALYSIS

Let Ω be a domain in $\mathbb{O}$, let $\mathbb{S} \subset \mathbb{O}$ be the imaginary sphere defined in (24) and let $\iota \in \mathbb{S}$. Following Gentili and Struppa in [8] we will now define a real differentiable function f from Ω to $\mathbb{O}$ as *regular* if for every $\iota \in \mathbb{S}$ its restriction f_ι to the complex line $L_\iota \cong \mathbb{R} \oplus \iota \mathbb{R}$ passing through the origin and containing 1 and

ι is holomorphic on $\Omega \cap L_\iota$. Moreover we will define the directional derivatives along ι as

$$\partial_\iota f(x + \iota y) := \frac{1}{2} \left(\frac{\partial}{\partial x} - \iota \frac{\partial}{\partial y} \right) f_\iota(x + \iota y), \quad (29)$$

$$\bar{\partial}_\iota f(x + \iota y) := \frac{1}{2} \left(\frac{\partial}{\partial x} + \iota \frac{\partial}{\partial y} \right) f_\iota(x + \iota y). \quad (30)$$

As in the complex case, f is regular if and only if $\bar{\partial}_\iota f(w) = 0$ for every $\iota \in \mathbb{S}$ and $w \in \Omega \cap L_\iota$. We then define

$$\partial f(w) = \begin{cases} \partial_\iota f(w) & \text{if } w = x + \iota y, y \neq 0, \\ \frac{\partial f}{\partial x}(x) & \text{if } w = x \in \mathbb{R}. \end{cases} \quad (31)$$

Series of results of complex analysis are then extended to the Octonions. It can be easily shown (see [8]) that the derivative of a regular function is a regular function, that

$$\partial^n f(x + \iota y) = \frac{\partial^n f}{\partial x^n}(x + \iota y), \quad (32)$$

that polynomials are regular functions and finally that a regular function has its series expansion

$$f(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0) w^n, \quad (33)$$

on a ball centered in the origin $B(0, R)$ where R is the radius of convergence of the series. The Liouville theorem equivalent also applies and if f is regular and $|f|$ has a relative maximum, then f is constant. Finally we also have the Cauchy representation formula for regular functions on Octonions, i.e.

$$f(w) = \frac{1}{2\pi\iota_w} \int_{\partial\Delta_w(0,r)} \frac{f(\zeta)}{\zeta - w} d\zeta, \quad (34)$$

where

$$\iota_w = \frac{\text{Im}(w)}{|\text{Im}(w)|}, \quad (35)$$

if $\text{Im}(w) \neq 0$ and $\Delta_w(0, r) = \{x + \iota_w y : x^2 + y^2 \leq r^2\}$.

3 THE EXCEPTIONAL JORDAN ALGEBRAS

Let A be a commutative non-associative algebra. If A satisfies the *Jordan identity*, i.e.

$$(ab)(aa) = a(b(aa)), \quad (36)$$

for all $a, b \in A$, then we say is a *Jordan algebra*. It can be shown[10] that A is *power-associative*, i.e.

$$(a^m b) a^n = a^m (b a^n), \quad (37)$$

and that the *formal reality* condition is valid, i.e.

$$a_1^2 + \dots + a_n^2 = 0 \implies a_1 = \dots = a_n = 0. \quad (38)$$

A well known result from Jordan, von Neumann and Wigner[10] states that every simple finite-dimensional formally real Jordan algebra is isomorphic to one of the following Jordan algebras:

- $\mathfrak{h}_n(\mathbb{R})$ the self adjoint real matrices with product $a \circ b = \frac{1}{2}(ab + ba)$;
- $\mathfrak{h}_n(\mathbb{C})$ the self adjoint complex matrices with product $a \circ b = \frac{1}{2}(ab + ba)$;
- $\mathfrak{h}_n(\mathbb{H})$ the self adjoint quaternion matrices with product $a \circ b = \frac{1}{2}(ab + ba)$;
- $\mathfrak{h}_3(\mathbb{O})$ the 3 by 3, self adjoint octonion matrices with product $a \circ b = \frac{1}{2}(ab + ba)$;
- $\mathbb{R}^n \oplus \mathbb{R}$ with product $(x, t) \circ (x', t') = (tx' + t'x, x \cdot x' + tt')$.

Obviously we say that a matrix is self-adjoint if the coefficient a_{ij} are equal to a_{ji}^* , using the conjugation defined on the division algebra used, i.e. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively.

It is worth noting that since the algebra of Octonions $\mathbb{O}$ is alternative we can have a Jordan algebra made by 3 by 3 matrices, but since is not associative, we cannot have $\mathfrak{h}_n(\mathbb{O})$ with n greater than 3. Therefore $\mathfrak{h}_2(\mathbb{O})$ and $\mathfrak{h}_3(\mathbb{O})$ are the only Jordan algebra that can be made with octonian self adjoint matrices.

The $\mathfrak{h}_2(\mathbb{O})$ algebra. The $\mathfrak{h}_2(\mathbb{O})$ algebra can be realized as the 2 by 2 hermitian matrices, i.e. matrices of the form

$$a = \begin{pmatrix} \alpha & x^* \\ x & \beta \end{pmatrix}, \quad (39)$$

where $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{O}$, equipped with the Jordan product

$$a \circ b = \frac{1}{2}(ab + ba), \quad (40)$$

based on the usual associative product over matrices. It will be shown in next section that this algebra is isomorphic to $\mathbb{R}^2 \oplus \mathbb{R}$ equipped with $(x, t) \circ (x', t') = (tx' + t'x, x \cdot x' + tt')$ and therefore doesn't appear in the classification theorem due to Jordan, von Neumann and Wigner.

The $\mathfrak{h}_3(\mathbb{O})$ algebra. The $\mathfrak{h}_3(\mathbb{O})$ algebra is the only exceptional Jordan algebra of finite dimension, specifically $\dim_{\mathbb{R}} \mathfrak{h}_3(\mathbb{O}) = 27$. It can be realized as 3 by 3 hermitian matrices, i.e. of the form

$$H = \begin{pmatrix} \alpha & x^* & z \\ x & \beta & y^* \\ z^* & y & \gamma \end{pmatrix}, \quad (41)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $x, y, z \in \mathbb{O}$ and x^*, y^*, z^* are their octonionic conjugates, equipped with the Jordan product. It is worth noting that, even if octonions are not associative nor commutative, it is possible to define the determinant function given by

$$\det(H) = \alpha\beta\gamma - \left(\alpha |x|^2 + \beta |y|^2 + \gamma |z|^2 \right) + 2\operatorname{Re}(xyz). \quad (42)$$

The geometry of the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ is rich and with numerous connections with exceptional Lie groups. It can be shown easily that the Lie group of the invertible linear transformation that preserve the determinant of the element of the algebra is the exceptional Lie group E_6 , therefore sometimes labeled as $SL(3, \mathbb{O})$; while if we also impose the preservation of the trace of the elements of the algebra we obtain the exceptional Lie group F_4 that is therefore also used as definition of $SU(3, \mathbb{O})$. For the complete listing of generators of $SL(3, \mathbb{O})$ and $SU(3, \mathbb{O})$ see [9], section 11.3 and 11.4.

4 POSSIBLE APPLICATIONS TO PARTICLE PHYSICS

The study of Octonions in particle physics have been started by Jordan back in 1933 and then by Jordan, von Neumann and Wigner in an early attempt to generalize quantum mechanics and formalize the algebra of observables[10]. In 1973 Gnaedinger and Gürsey proposed the use of octonions in a split basis to model the quark structure[6]; later on, Dixon used the algebra $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ to model some feature of particle physics[5]. In the first decade of the century, Dray and Manogue (see [9] and reference therein) continued to pursue the use of Octonions in particle physics in a series of articles; while Baez and Huerta [2][3] focused on the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$. More recently a series of papers by Todorov and Dubois-Violette[4], Krasnov[7] and Baez advanced on the role that $\mathfrak{h}_3(\mathbb{O})$ might have in the Standard Model. One of the main issues against the use of Octonions in Grand Unification Theories is the absence of a complex representation of groups related to Octonions such as G_2 and F_4 . This issue was recently addressed by Boyle in [12] introducing the complexification of $\mathfrak{h}_3(\mathbb{O})$ which we will briefly present in section 4.3.

4.1 MINKOWSKY SPACETIME FROM JORDAN ALGEBRAS

Let X be an element of $\mathfrak{h}_2(\mathbb{C})$ and let $\sigma_x, \sigma_y, \sigma_z$ be the Pauli matrices, i.e.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (43)$$

then we can consider an element (t, x, y, z) in the Minkowski space $\mathbb{M}^4$ as an element of $\mathfrak{h}_2(\mathbb{C})$ given by

$$X = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} = tI + x\sigma_x + y\sigma_y + z\sigma_z. \quad (44)$$

Since Lorentz transformation are those transformations that preserve the element

$$\det(X) = x^2 + y^2 + z^2 - t^2, \quad (45)$$

then we have the identification between $\text{Spin}(3, 1)$, i.e. the double cover of the Lorentz group $SO(3, 1)$, and

$$SL(2, \mathbb{C}) \cong \{\Lambda \in M_2(\mathbb{C}) : \det(\Lambda X \Lambda^\dagger) = \det(X), X \in \mathfrak{h}_2(\mathbb{C})\}, \quad (46)$$

where $\Lambda^\dagger$ is the transpose of Λ^* .

The same construction applies to all the division algebras $\mathbb{K}$, therefore we can identify the Jordan algebras $\mathfrak{h}_2(\mathbb{K})$ with the Minkowski spacetime of 3, 4, 6 and 10 dimension respectively, i.e.

$$\mathfrak{h}_2(\mathbb{R}) \cong \mathbb{M}^3, \quad (47)$$

$$\mathfrak{h}_2(\mathbb{C}) \cong \mathbb{M}^4, \quad (48)$$

$$\mathfrak{h}_2(\mathbb{H}) \cong \mathbb{M}^6, \quad (49)$$

$$\mathfrak{h}_2(\mathbb{O}) \cong \mathbb{M}^{10}. \quad (50)$$

On the other hand, the previous construction can also be used as a way to generalize the definition of $SL(2, \mathbb{K})$ and obtaining the identification

$$SL(2, \mathbb{R}) \cong \text{Spin}(2, 1), \quad (51)$$

$$SL(2, \mathbb{C}) \cong \text{Spin}(3, 1), \quad (52)$$

$$SL(2, \mathbb{H}) \cong \text{Spin}(5, 1), \quad (53)$$

$$SL(2, \mathbb{O}) \cong \text{Spin}(9, 1). \quad (54)$$

4.2 OCTONIONIC REPRESENTATION OF $SU(3)$

One of the main interest for the use of Octonions in particle physics was motivated by the realization of $SU(3)$ as subgroup of the exceptional Lie group G_2 , as was pointed out in [6]. We have already shown that the group of automorphisms of $\mathbb{O}$ is the compact form of G_2 . If we choose an imaginary unit in $\mathbb{O}$, the group of automorphisms of $\mathbb{O}$ that fixes the imaginary unit is indeed isomorphic to $SU(3)$.

The Lie group

$$SU(3) = \{X \in M_n(\mathbb{C}) : X^\dagger X = I, \det(X) = 1\} \quad (55)$$

is relevant for quantum chromodynamics since a gauge colour rotation is a $SU(3)$ group element $U(x)$ obtained through the exponential map, i.e.

$$U(x) = \exp\left(i \sum_{k=1}^8 \theta^k(x) \lambda_k\right), \quad (56)$$

where $x \in \mathbb{M}^4$ and the λ_k are generators for the Lie algebra $\mathfrak{su}(3)$ and are called the *Gell-Mann matrices*, i.e.

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{57}$$

Using the Gell-Mann matrices we can have a 14-dimensional representation for G_2 as

$$\begin{aligned}
T^k &= \frac{1}{2\sqrt{2}} \begin{pmatrix} \lambda_k & 0 & 0 \\ 0 & -\lambda_k^* & 0 \\ 0 & 0 & 0 \end{pmatrix}, k = 1, \dots, 8, \\
T^9 &= \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & -i\lambda_7 & \sqrt{2}e_1 \\ i\lambda_7 & 0 & \sqrt{2}e_1 \\ \sqrt{2}e_1^\dagger & \sqrt{2}e_1^\dagger & 0 \end{pmatrix}, T^{10} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & -\lambda_7 & i\sqrt{2}e_1 \\ -\lambda_7 & 0 & -i\sqrt{2}e_1 \\ -i\sqrt{2}e_1^\dagger & i\sqrt{2}e_1^\dagger & 0 \end{pmatrix}, \\
T^{11} &= \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & i\lambda_5 & \sqrt{2}e_2 \\ -i\lambda_5 & 0 & \sqrt{2}e_2 \\ \sqrt{2}e_2^\dagger & \sqrt{2}e_2^\dagger & 0 \end{pmatrix}, T^{12} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & \lambda_5 & i\sqrt{2}e_2 \\ \lambda_5 & 0 & -i\sqrt{2}e_2 \\ -i\sqrt{2}e_2^\dagger & i\sqrt{2}e_2^\dagger & 0 \end{pmatrix}, \\
T^{13} &= \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & -i\lambda_2 & \sqrt{2}e_3 \\ i\lambda_2 & 0 & \sqrt{2}e_3 \\ \sqrt{2}e_3^\dagger & \sqrt{2}e_3^\dagger & 0 \end{pmatrix}, T^{14} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & -\lambda_2 & i\sqrt{2}e_3 \\ -\lambda_2 & 0 & -i\sqrt{2}e_3 \\ -i\sqrt{2}e_3^\dagger & i\sqrt{2}e_3^\dagger & 0 \end{pmatrix},
\end{aligned} \tag{58}$$

where we used e_1, e_2 , and e_3 to represent the standard orthonormal vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{59}$$

Using the above representation of G_2 it is straightforward to identify $SU(3)$ as the subgroup generated by $T^1, \dots, T^8$.

It is worth noting that the above representation of G_2 , and consequently of $SU(3)$, is isomorphic to its complex conjugate representation. This is a necessary condition for the representation to be *real* and all representations of G_2 , along with other exceptional Lie groups such as F_4, E_7 and E_8 , only have real representations.

4.3 THE STANDARD MODEL GAUGE GROUP G_{SM} AND $\mathfrak{h}_3(\mathbb{O})$

The Standard Model was structured on the works of Glashow, Weinberg, Salam and Higgs as a gauge theory constituted by a product of three different groups,

each responsible for a different type of interaction, i.e.

$$SU(3) \times SU(2) \times U(1), \quad (60)$$

where $SU(3)$ is for strong interactions, $SU(2)$ accounts for the weak interaction, while $U(1)$ is the hypercharge group. Even if extremely successful, the Standard Model leaves open a lot of basic questions such as the origin of its gauge group. In fact, since the Standard Model has an additional symmetry given by a group isomorphic to $\mathbb{Z}_6$, the true Standard Model gauge group is defined as

$$G_{SM} = [SU(3) \times SU(2) \times U(1)] / \mathbb{Z}_6. \quad (61)$$

Now we will show how the gauge group G_{SM} can be obtained as the intersection of two F_4 subgroup and therefore resulting from automorphisms of the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$.

Let $w \in \mathbb{O}$ and consider, let $\mathbb{S} \subset \mathbb{O}$ be the imaginary sphere defined in (24) and let $\iota \in \mathbb{S}$. The decomposition

$$w = (x_0 + \iota x_1) + (x_2 + \iota x_3)i + (x_4 + \iota x_5)j + (x_6 + \iota x_7)k, \quad (62)$$

where $\iota, i, j, k \in \mathbb{S}$ and $x_0, \dots, x_7 \in \mathbb{R}$, establishes a map δ between the Octonions $\mathbb{O}$ and $\mathbb{C} \oplus \mathbb{C}^3$ given by

$$\delta : w \longrightarrow z + Z, \quad (63)$$

where $z = x_0 + \iota x_1 \in \mathbb{C}$ and $Z = (x_2 + \iota x_3, x_4 + \iota x_5, x_6 + \iota x_7)^t \in \mathbb{C}^3$. We have already shown that elements of $\mathfrak{h}_3(\mathbb{O})$ are of the form

$$H = \begin{pmatrix} \alpha & w_1^* & w_3 \\ w_1 & \beta & w_2^* \\ w_3^* & w_2 & \gamma \end{pmatrix}, \quad (64)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $w_k \in \mathbb{O}$ and w_k^* are their octonionic conjugates. Therefore, applying δ to w_k we can associate to every element H of $\mathfrak{h}_3(\mathbb{O})$ an element $h + m$ of $\mathfrak{h}_3(\mathbb{C}) \oplus M_3(\mathbb{C})$ given by

$$h = \begin{pmatrix} \alpha & z_1^* & z_3 \\ z_1 & \beta & z_2^* \\ z_3^* & z_2 & \gamma \end{pmatrix}, m = (Z_1, Z_2, Z_3), \quad (65)$$

where $z_k \in \mathbb{C}$, z_k^* are their complex conjugates and therefore $h \in \mathfrak{h}_3(\mathbb{C})$, while $Z_k \in \mathbb{C}^3$ and therefore $m \in M_3(\mathbb{C})$. The general automorphism of $\mathfrak{h}_3(\mathbb{O})$ that preserves the previous embedding [13] is given by

$$h \longrightarrow VhV^\dagger, m \longrightarrow UhV^\dagger, \quad (66)$$

where $U, V \in SU(3)$ and since the automorphism is unchanged under $e^{2\pi i/3}$ multiplication then they form a subgroup of F_4 isomorphic to $[SU(3) \times SU(3)] / \mathbb{Z}_3$.

We have required that the automorphisms of $\mathfrak{h}_3(\mathbb{C})$ preserved the chosen $\mathbb{C} \oplus \mathbb{C}^3$ decomposition of $\mathcal{O}$. In [4] Todorov and Dubois-Violette showed that choosing a rank one idempotent on $\mathfrak{h}_3(\mathbb{C})$ such as

$$\Pi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (67)$$

and imposing the preservation of the idempotent, automorphisms in (66) would then require an additional feature on $V \in SU(3)$, i.e. being of the form

$$V = \begin{pmatrix} \varphi \tilde{V} & 0 \\ 0 & \varphi^{-2} \end{pmatrix}, \quad (68)$$

where $\tilde{V} \in SU(2)$ and $\varphi \in U(1)$ and therefore the whole automorphism would be determined by

$$(U, \tilde{V}, \varphi) \in SU(3) \times SU(2) \times U(1). \quad (69)$$

To conclude it is worth noting that the automorphism results unchanged with the following transformations

$$(U, \tilde{V}, \varphi) \longrightarrow (U, -\tilde{V}, -\varphi), \quad (70)$$

$$(U, \tilde{V}, \varphi) \longrightarrow (e^{2\pi i/3}U, \tilde{V}, e^{2\pi i/3}\varphi), \quad (71)$$

and therefore the resulting subgroup of automorphisms is indeed the true gauge group, i.e.

$$SU(3) \times SU(2) \times U(1) / \mathbb{Z}_6 < \text{Aut}(\mathfrak{h}_3(\mathcal{O})). \quad (72)$$

4.4 COMPLEXIFICATION $\mathfrak{h}_3^{\mathbb{C}}(\mathcal{O})$ AND FERMIONS REPRESENTATION

The fact that F_4 does not have any complex representation, i.e. a representation where the complex conjugate is a non-equivalent representation, has prevented pursuing the search of Grand Unified Theories with F_4 as gauge group, along with G_2, E_7 and E_8 . Nevertheless it is worth noting[13] that the automorphisms of complexification of the exceptional Jordan Algebra $\mathfrak{h}_3(\mathcal{O})$, i.e. $\mathfrak{h}_3^{\mathbb{C}}(\mathcal{O})$, form the compact group E_6 which indeed has complex representation, i.e.

$$\text{Aut}(\mathfrak{h}_3^{\mathbb{C}}(\mathcal{O})) \cong E_6. \quad (73)$$

It is well known that E_6 is the only exceptional group to have complex representation and therefore it has been a natural candidate as gauge group in Grand Unified Theories along with $SU(5)$ and $SO(10)$ (e.g. [14, 15]).

We can therefore reproduce the previous construction on $\mathfrak{h}_3^{\mathbb{C}}(\mathbb{O})$. Let $\iota_1 \in \mathbb{S}$ be the imaginary unit used to decompose $\mathbb{O}$ as in the previous section, ι_2 the imaginary unit for the complex scalar field in $\mathbb{C} \otimes \mathfrak{h}_3(\mathbb{O})$ and

$$\iota = \frac{1}{2}(1 + \iota_1 \iota_2). \quad (74)$$

Then, the general automorphism of $\mathfrak{h}_3^{\mathbb{C}}(\mathbb{O})$ that preserves the embedding of $\mathbb{C}$ in $\mathbb{O}$ is given by

$$h \longrightarrow V_{LR} h V_{LR}^\dagger, m \longrightarrow U h \bar{V}_{LR}^\dagger, \quad (75)$$

where $z \longrightarrow \bar{z}$ is the conjugation in respect to the scalar field $\mathbb{C}$,

$$V_{LR} = V_L \iota + V_R \bar{\iota}, \quad (76)$$

and $U, V_L, V_R \in SU(3)$. Then, if we impose, as in the previous section, an idempotent to be invariant, we then have the decomposition

$$V_L = \begin{pmatrix} \varphi \tilde{V}_L & 0 \\ 0 & \varphi^{-2} \end{pmatrix}, V_R = \begin{pmatrix} \varphi \tilde{V}_R & 0 \\ 0 & \varphi^{-2} \end{pmatrix}, \quad (77)$$

with $\tilde{V}_L, \tilde{V}_R \in SU(2)$, leading to the identification with the gauge group of the minimal left-right symmetric extension of the Standard Model, i.e.

$$G_{LR} = [SU(3) \times SU(2)_L \times SU(2)_R \times U(1)] / \mathbb{Z}_6. \quad (78)$$

4.5 CONCLUSIONS

The gauge group of the Standard Model can be obtained considering the automorphisms of the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ that fix a complex line in the Octonions and another complex line in the remaining $\mathbb{C}^3$ structure. As Baez evidenced, this is equivalent to say that the gauge group G_{SM} is obtained from $\mathfrak{h}_3(\mathbb{O})$ picking a 10-D Minkowski space and imposing its invariance, and then inside of it imposing the invariance of a 4D Minkowski space. Similarly the gauge group of the minimal left-right symmetric extension of the Standard Model G_{LR} is obtained from $\mathfrak{h}_3^{\mathbb{C}}(\mathbb{O})$ picking a complex line in the Octonions and another complex line in the remaining $\mathbb{C}^3$ structure. The latter construction has the property of having a complex representation that therefore is suitable to be a real candidate for Grand Unified Theories.

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