SOR-like methods for solving the Sylvester equation

Jakub Kierzkowski

Faculty of Mathematics and Information Science
Warsaw University of Technology

J.Kierzkowski@mini.pw.edu.pl

Abstract

We present the **SOR-like methods** and highlight some of their known properties. We give the SOR-like method as proposed by Z. Woźnicki and propose two similar methods based upon it. All three are stationary iterative methods for solving the Sylvester equation ($AX - XB = C$). We form two sufficient conditions under which one of those methods will converge. In addition, we present a modification method, based on the following fact: if X is a solution of $AX - XB = C$, it is also a solution of $(A - \alpha I_m)X - X(B - \alpha I_n) = C$. We also present numerical experiments to illustrate the theoretical results and some properties of the methods.

1. Introduction

Definition 1. Let $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times n}$ be given matrices, and $X \in \mathbb{R}^{m \times n}$ be an unknown matrix. The Sylvester equation then is the equation of the form

$$AX - XB = C.$$

The **Sylvester equation** has many applications, for example in control theory and when solving partial differential equations numerically.

Let A be cut into parts $A = D - L - U$, where L and U are lower and upper triangular, with zeros on the main diagonals, respectively, and D is nonsingular diagonal:

Algorithm 1: SOR-like method

$$X^{(t)} = (I - \omega D^{-1}L)^{-1} \left(((1 - \omega)I + \omega D^{-1}U) X^{(t-1)} + \omega D^{-1} (X^{(t-1,t)}B + C) \right)$$

The most characteristic idea in this method is the usage of the double index $(t-1, t)$ – we take the entries of the latest approximation $X^{(t)}$ if they are already computed in the t -th iteration, and entries of the previous approximation $X^{(t-1)}$ for all the rest of the entries.

To avoid such difficulties, we can modify the method's formula by replacing the double index $(t-1, t)$ with the simple $(t-1)$:

Algorithm 2: ISOR-like method

$$X^{(t)} = (I - \omega D^{-1}L)^{-1} \left(((1 - \omega)I + \omega D^{-1}U) X^{(t-1)} + \omega D^{-1} (X^{(t-1)}B + C) \right)$$

On the other hand, if there already is the double index in the formula, it can be used in all occurrences of the matrix X , hoping that it will improve the convergence:

Algorithm 3: bSOR-like method

$$X^{(t)} = (I - \omega D^{-1}L)^{-1} \left(((1 - \omega)I + \omega D^{-1}U) X^{(t-1,t)} + \omega D^{-1} (X^{(t-1,t)}B + C) \right)$$

If X is a solution of $AX - XB = C$, it is also a solution of $(A - \alpha I_m)X - X(B - \alpha I_n) = C$ for all α . Any of the SOR-like methods can be used to solve the modified problem with $\alpha \neq 0$. This can be also treated in a different way, as using a different (more general) method for the basic problem $AX - XB = C$, where α is a method's parameter, like ω is. We call this approach the **α SOR-like methods**. Any SOR-like method is a special case of its corresponding α SOR-like method for $\alpha = 0$.

Algorithm 4: α ISOR-like method

$$X^{(t)} = (I - \omega(D - \alpha I_m)^{-1}L)^{-1} \cdot \left(((1 - \omega)I + \omega(D - \alpha I_m)^{-1}U) X^{(t-1)} + \omega D^{-1} (X^{(t-1)}B - \alpha X^{(t-1)} + C) \right)$$

α SOR-like and α bSOR-like methods are defined the same way.

2. ISOR-like convergence

Let $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times n}$, $A \neq 0$, $B \neq 0$ and $\det(D) \neq 0$.

Theorem 1 (First sufficient condition for ISOR-like convergence).

Let $G = \|D^{-1}L\| (1 + \|D^{-1}U\| + \|D^{-1}\| \cdot \|B\|)$, $H = 1 + \|D^{-1}U\| + \|D^{-1}\| \cdot \|B\| - \|D^{-1}L\|$, $M = \|D^{-1}L\| (-1 + \|D^{-1}U\| + \|D^{-1}\| \cdot \|B\|)$, $N = -1 + \|D^{-1}U\| + \|D^{-1}\| \cdot \|B\| + \|D^{-1}L\|$.

Then if $\|I - \omega D^{-1}L\| \geq \|(I - \omega D^{-1}L)^{-1}\|$ and

$$\begin{cases} G \cdot \omega^2 + H \cdot \omega - 2 < 0 \text{ for } \omega \geq 1 \\ M \cdot \omega^2 + N \cdot \omega < 0 \text{ for } 0 < \omega \leq 1 \end{cases}$$

then ISOR-like converges.

Note that for $\omega = 1$, $G \cdot \omega^2 + H \cdot \omega - 2 = M \cdot \omega^2 + N \cdot \omega$.

Theorem 2 (Second sufficient condition for ISOR-like convergence).

If $\omega > 0$ and

$$\frac{|1 - \omega| + \omega \|D^{-1}U\| + \omega \|D^{-1}\| \|B\|}{1 - \omega \|D^{-1}L\|} \in (0; 1)$$

then ISOR-like converges.

3. Numerical tests

3.1 Example 1. Numerical solution for Laplace's PDE

$-\Delta u = 2 \sin(y)(x \sin(x) - \cos(x))$, $u|_{\partial U} = 0$, $U = [0, \pi] \times [0, 2\pi]$, $A \in \mathbb{R}^{12 \times 12}$, $B \in \mathbb{R}^{25 \times 25}$,

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & \ddots & \vdots \\ 0 & 0 & \dots & 2 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & \ddots & \vdots \\ 0 & 0 & \dots & -2 \end{bmatrix}$$

$\alpha = [-11, -10.5, -10, \dots, 1.5, 2, 2.5]$, $\omega = [0.1, 0.2, 0.3, \dots, 4.8, 4.9, 5]$, $C \in \mathbb{R}^{12 \times 25}$ is from the right hand side function, initial guess $X^{(0)} = C$.

Table 1: α and ω values for which the methods converged the fastest

method	α_{opt}	ω_{opt}	iterations
α ISOR-like	-6.5	2.8	750*
α SOR-like	-2	1.7	89
α bSOR-like	-5	2.9	311

Figure 1: (α, ω) pairs for which ISOR-like converged

Figure 2: Numbers of iterations for (α, ω) pairs

The conditions of Theorem 1 were not satisfied for any of the tested values. Among the tested values, the conditions of Theorem 2 were satisfied for $\alpha = [-11, -10.5, \dots, -2]$ and $\omega = [0.1, 0.2, \dots, 1]$. All the SOR-like methods did converge for that values.

3.2 Example 2.

For given α and ω , using the Kronecker product and the vectorization operation (see eg. [1]), Algorithm 4. can be written in the form $\vec{X}^{(t)} = M \vec{X}^{(t-1)} + \vec{Z}$, where M is a transition matrix. In this example we present the value of the spectral radius of M depending on α and ω .

$$A = \begin{bmatrix} 10 & 1 & \dots & 1 \\ 1 & 11 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 19 \end{bmatrix}, B = \begin{bmatrix} 0.382 & 0.382 \\ 0.382 & 0.764 \end{bmatrix}$$

Figure 3: Spectral radius of M . White color for values greater than 1.

The α ISOR method does not converge for $\alpha \in (10, 19)$, where 10 and 19 are the minimum and maximum of the diagonal of A . More, for big enough α , the method converge for $\omega < 0$.

References

- [1] J.Kierzkowski, *New SOR-like methods for solving the Sylvester equation*. Open Mathematics 13(1), pages 178-187, 2015.
- [2] Z. Woźnicki, *Solving Linear Systems: An Analysis of Matrix Prefactorization Iterative Methods*. Matrix Editions, 2009.