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MAXIMAL PRIME GAPS BOUNDS

JAN FELIKSIK

ABSTRACT.

This paper presents research results, pertinent to the maximal prime gaps bounds. Four distinct bounds are presented: Upper bound, Infimum, Supremum and finally the Lower bound. Although the Upper and Lower bounds incur a relatively high estimation error cost, the functions representing them are quite simple. This ensures, that the computation of those bounds will be straightforward and efficient.

The Lower Bound, is essential to address the issue of the value of the lower bound constant $\mathcal{C}$, in the work of Ford et al [16]:

$$p_{(n+1)} - p_{(n)} \gg \mathcal{C} \frac{(\log N) (\log \log N) (\log \log \log N)}{\log \log \log N}$$

The concluding Corollary in this paper, shows that the value of the constant $\mathcal{C}$ does diverge, although slowly. The constant $\mathcal{C}$, will eventually take any arbitrary value, providing that a large enough N (for $p_{(n+1)} \leq N$) is considered.

The Infimum/Supremum bounds on the other hand are computationally very demanding. Their evaluation entails computations at an extreme level of precision. In return however, we obtain bounds, which provide an extremely close approximation of the maximal prime gaps. The Infimum/Supremum estimation error gradually increases over the range of p_n , and attains at $p_n = 18361375334787046697$ approximately the value of 0.03.

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Key words and phrases. Distribution of primes, Ford-Green-Konyagin-Maynard-Tao maximal gaps problem, maximal prime gaps Lower bound, maximal prime gaps Upper bound, maximal prime gaps Infimum, maximal prime gaps Supremum, Prime Number Theorem, Erdős - Rankin maximal prime gap problem.

1. UPPER BOUND ON MAXIMAL PRIME GAPS FROM HISTORICAL PERSPECTIVE

The topic of the upper bound on the maximal prime gaps, had been discussed in Feliksiak [14] research paper, where the Upper Bound on the Maximal Prime Gaps is shown to be:

Theorem 1.1 (Upper Bound on Maximal Prime Gaps).

The Upper Bound on Maximal Prime Gaps, for all $p_n \in \mathbb{N} \mid p_n \geq 11$ is given by:

$$(1.1) \quad \mathfrak{g} = (p_{n+1} - p_n) \leq 5 (\log_{10} p_n)^2 = \mathcal{UB}_{(p_n)}$$

Proof.

For a proof please refer to: pp. 8, Feliksiak [14]. □

However, already in 1936 a Swedish mathematician Harald Cramér, attempted to formulate the maximal prime gaps upper bound, by implementing a sophisticated heuristic argument. He argued, that there exists at least one prime $p \in \mathbb{N}$, within the interval:

$$(1.2) \quad n < p \leq \left(n + (\log p)^2 \right)$$

From the Cramér's model we have:

$$(1.3) \quad \lim_{n \rightarrow \infty} \left(\frac{p_{(n+1)} - p_{(n)}}{(\log p_{(n)})^2} \right) = \alpha$$

H. Cramér asserted that $\alpha = 1$. Over the years, the Cramér's Maximal Gaps Conjecture had been considered too strong. In 1995 A. Granville proposed, that there is a bound M for which:

$$\lim_{n \rightarrow \infty} \left(\frac{p_{(n+1)} - p_{(n)}}{(\log p_{(n)})^2} \right) < M$$

It had been suggested, that $M = 2e^{-\gamma} \approx 1.12291897$. However, substituting from Theorem 1.1 into the Cramér's model, we obtain:

$$(1.4) \quad \alpha = \lim_{n \rightarrow \infty} \left(\frac{p_{(n+1)} - p_{(n)}}{(\log p_{(n)})^2} \right) = \frac{5}{(\log 10)^2} \approx 0.9430585$$

H. Riesel [25], came up with a test for predictions of the occurrence of maximal gaps of a given size, based on the Cramér's model :

$$\mathcal{R} = \frac{\log p_{(n)}}{\sqrt{\mathfrak{g}}}$$

With the limit expected to be:

$$\lim_{p_n, \mathfrak{g} \rightarrow \infty} (\mathcal{R}) \rightarrow 1$$

From the empirical evidence obtained for the values of $\mathcal{R}$, it seems reasonable:

<i>prime</i>	<i>gap size</i>	$\mathcal{R}$
23	5	1.40224
1129	21	1.53387
370261	111	1.21701
4302407359	353	1.18065
171231342420521	805	1.15513
218209405436543	905	1.0975
1425172824437699411	1475	1.0884

However, implementing Theorem 1.1, we determine this limit to be:

$$(1.5) \quad \lim_{p_n, \mathfrak{g} \rightarrow \infty} (\mathcal{R}) = \lim_{p_n \rightarrow \infty} \left(\frac{\log p_{(n)}}{\sqrt{5 (\log_{10} p_{(n)})^2}} \right) = \frac{\log 10}{\sqrt{5}} \approx 1.02975$$

2. THE MAXIMAL PRIME GAPS INFIMUM/SUPREMUM BOUNDS

The Infimum/Supremum bounds on the maximal prime gaps, implement Riemann Sums with regular subdivisions of 250000 per unit. This means that over the prime gap, there are $\mathfrak{g} \times 250000$ subdivisions. This was done to overcome technical issues, while evaluating the definite logarithmic integral. All necessary computations were carried out with the aid of *Mathematica*® software, with numerical precision set to 300000 decimal places.

Maximal Prime Gaps vs Complete Set Of Bounds

Function Value

FIGURE 1. The drawing shows the graph of the Maximal Prime Gaps (Blue), the Infimum/Supremum Bounds (Red) and Lower/Upper Bounds (Black). The figure is drawn w.r.t. ξ (Def. 3.1), in the range $p_n \in \mathbb{N} \mid 23 \leq p_n \leq 18\,361\,375\,334\,787\,046\,697$.

Remark 2.1.

The Upper/Lower bounds shown in Figure 1, are given by Theorem 1.1 and Theorem 3.4 respectively. **The Auxiliary Infimum/Supremum bounds, presented in the Figure 1, were generated purely for visualization purposes.** They are the direct consequence of Theorems 2.6 and 2.7, which specify bounds that are profoundly superior to them. Since for technical reasons, it is impossible to draw the Infimum/Supremum bounds given by Theorems 2.6 and 2.7 vs the Maximal Prime Gaps curve, consequently, only the Infimum/Supremum estimation error graphs are presented. Please refer to Figures 2 and 4. The Auxiliary Infimum/Supremum graphs presented in the Figure 1, can be generated by implementing the Riemann Sum 2.1:

$$(2.1) \quad C_1 \left(\frac{(\log_{10} p_n)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limt-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] + C_2$$

With the constants (Auxiliary Infimum) defined by:

$$C_1 = \frac{959758\pi}{1000000}, \alpha = 1, \beta = 1.020 \text{ and } C_2 = (-1)$$

With the constants (Auxiliary Supremum) defined by:

$$C_1 = \frac{9359\pi}{10000}, \alpha = 1, \beta = 1.00250 \text{ and } C_2 = 0.05$$

and implementing Definitions 2.1 through 2.5. The estimation error at the last maximal prime gap at $p_n = 18361375334787046697$, in the case of the Auxiliary Infimum is ≈ 35.328 (with a global min. at $p_n = 1327 \approx 0.58286$), while in the case of the Auxiliary Supremum it is ≈ 35.3281 . Both bounds are defined from $p_n = 23$.

First, we make a few general definitions valid within the entire section. In this section, the Riemann sums are implemented, with regular partitioning of the maximal prime gap interval. The number of regular divisions, between any two consecutive integers equals 250 000. Maximal Prime Gap of a size $\mathfrak{g}$, is a gap that strictly exceeds in size, any prime gap that precedes it. Let p_i and p_c denote the initial prime at which the Maximal Prime Gap begins and the concluding prime at which the Maximal Prime Gap ends, respectively.

Definition 2.1 (Maximal Prime Gap).

$$\mathfrak{g} = p_c - p_i$$

Definition 2.2 (Number of divisions of a Regular Partition).

$$d = 250000$$

Definition 2.3 (Summation Limit).

$$limt = d \times \mathfrak{g}$$

Definition 2.4.

$$\Delta x = \frac{\mathfrak{g}}{limt} = \frac{1}{d}$$

Definition 2.5 (Initial Element of Summation).

$$x_1^* = p_i + \left(\frac{1}{2d}\right)$$

Difference: Infimum Bound and True Gaps

FIGURE 2. The drawing shows the graph of the estimation error made by the Infimum Bound. The figure is drawn w.r.t. p_n , in the range $p_n \in \mathbb{N} \mid 4652353 \leq p_n \leq 18361375334787046697$.

Theorem 2.6 (Maximal Prime Gaps Infimum).

The Infimum Bound on the Maximal Prime Gaps, for $p_i \in \mathbb{N} \mid p_i \geq 23$ is given by:

$$(2.2) \quad \mathcal{INF}_{p_i} = C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right]$$

With the constants:

$$C_1 = \frac{907727\pi}{1000000}, \alpha = 1.00187651 \text{ and } \beta = 1.00187877$$

Proof.

The Riemann sum 2.2 approximates the logarithmic integral over the maximal prime gaps:

$$(2.3) \quad C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \int_{p_i}^{p_c} \left(\frac{1}{(\log t)^\beta} \right) dt \\ \approx C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right]$$

Let's denote a maximal prime gap:

$$(2.4) \quad \mathcal{G}_N = \max_{p_c \leq N} (p_c - p_i)$$

In 1931 Westzynthius [31] proved,

$$(2.5) \quad \lim_{p_c \rightarrow \infty} \left(\frac{\mathcal{G}_N}{\log p_c} \right) \rightarrow \infty$$

Equivalently, **in the case of the maximal prime gaps**:

$$(2.6) \quad \lim_{p_c \rightarrow \infty} \left(\frac{p_c - p_i}{\log p_c} \right) = \lim_{p_c \rightarrow \infty} \left(\frac{p_c - p_i}{(\log p_c)^\beta} \right) \rightarrow \infty$$

By the PNT for all $p_i \in \mathbb{N} \mid p_i \geq 23$ we may estimate the logarithmic integral, hence the Riemann's Sum:

$$(2.7) \quad \frac{p_c - p_i}{(\log p_c)^\beta} \lesssim \int_{p_i}^{p_c} \frac{1}{(\log t)^\beta} dt \lesssim \sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x$$

Clearly,

$$\lim_{p_i \rightarrow \infty} C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \rightarrow \infty$$

Consequently, we have:

$$(2.8) \quad \begin{aligned} & \lim_{p_i \rightarrow \infty} \left(C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \int_{p_i}^{p_c} \left(\frac{1}{(\log t)^\beta} \right) dt \right) \\ &= \lim_{p_i \rightarrow \infty} \left(C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] \right) \rightarrow \infty \end{aligned}$$

Suppose that Theorem 2.6 is false for some $p_i \in \mathbb{N} \mid p_i \geq 4652353$. Define now:

$$(2.9) \quad \mathcal{MPG}_{p_i} = \left(\mathfrak{g} - \frac{\log \log \log p_i}{330} \right) \quad \forall p_i \in \mathbb{N} \mid p_i \geq 4652353$$

The natural logarithm function $(\log \log \log p_i) / 330$ for $p_i \geq 4652353$, is a positive, increasing function, which at $p_i = 4652353$ attains approx. 0.00304478 and diverges. Thus, in accordance with the hypothesis, **at the maximal gaps** we must have:

$$(2.10) \quad \mathfrak{g} - C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] < 0$$

Since $\mathcal{MPG}_{p_i} < \mathfrak{g} \quad \forall p_i \geq 4652353$, necessarily therefore:

$$(2.11) \quad \begin{aligned} & \mathcal{MPG}_{p_i} - C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] \\ &= (\mathfrak{g} - \mathcal{NF}_{p_i}) - \frac{\log \log \log p_i}{330} = (\text{est. error}) - \frac{\log \log \log p_i}{330} < 0 \end{aligned}$$

Thus necessarily, the ratio a_j must be:

$$(2.12) \quad a_j = \frac{\text{estimation error}}{(\log \log \log p_i) / 330} = \frac{\left(\mathfrak{g} - C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5}-1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] \right)}{((\log \log \log p_i) / 330)} < 1$$

where j is the Maximal Gap ordinal number. The ratio at $j = 20$, $p_i = 4652353$, attains approx. 1.01267 and strictly increases at every step j , diverging as j increases. Please refer to Figure 3. Consequently this implies, that we have a contradiction to the initial assumption. This implies that:

$$(2.13) \quad \lim_{p_i \rightarrow \infty} \left(\mathfrak{g} - C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5}-1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] \right) \rightarrow \infty$$

which in turn implies, that Theorem 2.6 holds for $p_i \in \mathbb{N} \mid p_i \geq 4652353$. For $23 \leq p_i \leq 4652353$, direct computer evaluation demonstrates that, Theorem 2.6 holds in this range as well. Please refer to Table 2. Hence, Theorem 2.6, holds as stated for all $p_i \in \mathbb{N} \mid p_i \geq 23$. Concluding the proof. $\square$

FIGURE 3. The drawing shows ratio 2.12. The log-linear figure is drawn w.r.t. $p_i \in \mathbb{N}$ s.t. $4652353 \leq p_i \leq 18361375334787046697$.

Theorem 2.7 (Maximal Prime Gaps Supremum).

The Supremum Bound on the Maximal Prime Gaps, for $p_n \in \mathbb{N} \mid p_n \geq 155921$ is given by:

$$(2.14) \quad C_1 \left(\frac{(\log_{10} p_n)^\alpha}{(\sqrt{5}-1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right]$$

Difference: Supremum Bound and True Gaps

FIGURE 4. This log-linear drawing shows the graph of the estimation error made by the Supremum Bound. Please refer to Table 3 in the Appendix. The figure is drawn w.r.t. $p_n \in \mathbb{N}$, in the range $155921 \leq p_n \leq 18361375334787046697$.

With the constants:

$$C_1 = \frac{907765\pi}{1000000}, \quad \alpha = 1.00187651 \text{ and } \beta = 1.00187874$$

Remark 2.2. The range of the Supremum Bound may be extended, to include all primes $p_n \geq 523$ by adding the constant 0.0592 and re-computing pertinent data.

Proof.

The Riemann sum 2.14 approximates the logarithmic integral over the maximal prime gaps:

$$(2.15) \quad C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \int_{p_i}^{p_c} \left(\frac{1}{(\log t)^\beta} \right) dt \\ \approx C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limt-1)} \left(\frac{1}{(\log (x_1^* + k\Delta x))^\beta} \right) \Delta x \right]$$

Let's denote a maximal prime gap:

$$(2.16) \quad \mathcal{G}_N = \max_{p_c \leq N} (p_c - p_i)$$

In 1931 Westzynthius [31] proved,

$$(2.17) \quad \lim_{p_c \rightarrow \infty} \left(\frac{\mathcal{G}_N}{\log p_c} \right) \rightarrow \infty$$

Equivalently, **in the case of the maximal prime gaps**:

$$(2.18) \quad \lim_{p_c \rightarrow \infty} \left(\frac{p_c - p_i}{\log p_c} \right) = \lim_{p_c \rightarrow \infty} \left(\frac{p_c - p_i}{(\log p_c)^\beta} \right) \rightarrow \infty$$

By the PNT for all $p_i \in \mathbb{N} \mid p_i \geq 155921$ we may estimate the logarithmic integral, hence the Riemann's Sum:

$$\frac{p_c - p_i}{(\log p_c)^\beta} \lesssim \left[\int_{p_i}^{p_c} \frac{1}{(\log t)^\beta} dt \right] \lesssim \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right]$$

Clearly,

$$\lim_{p_i \rightarrow \infty} C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \rightarrow \infty$$

Consequently, we have:

$$(2.19) \quad \lim_{p_i \rightarrow \infty} \left(C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \int_{p_i}^{p_c} \left(\frac{1}{(\log t)^\beta} \right) dt \right) \\ = \lim_{p_i \rightarrow \infty} \left(C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] \right) \rightarrow \infty$$

Suppose that Theorem 2.7 is false for some $p_i \in \mathbb{N} \mid p_i \geq 1357201$. Define now:

$$(2.20) \quad \mathcal{MPG}_{p_i} = \left(\mathfrak{g} + \frac{\log \log \log p_i}{380} \right) \quad \forall p_i \in \mathbb{N} \mid p_i \geq 1357201$$

The natural logarithm function $(\log \log \log p_i) / 380$ for $p_i \geq 1357201$, is a positive, increasing function, which at $p_i = 1357201$ attains approx. 0.0025623 and diverges. Thus, in accordance with the hypothesis, **at the maximal gaps** we must have:

$$(2.21) \quad C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] - \mathfrak{g} < 0$$

Since $\mathcal{MPG}_{p_i} > \mathfrak{g} \quad \forall p_i \in \mathbb{N} \mid p_i \geq 1357201$, necessarily therefore:

$$(2.22) \quad C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] - \mathcal{MPG}_{p_i} \\ = (\mathcal{SUP} - \mathfrak{g}) - (\log \log \log p_i) / 380 = (\text{estimation error}) - (\log \log \log p_i) / 380 < 0$$

Thus necessarily, the ratio must be:

$$(2.23) \quad a_j = \frac{(\text{estimation error})}{(\log \log \log p_i) / 380} \\ = \frac{\left(C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(limit-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] - \mathfrak{g} \right)}{((\log \log \log p_i) / 380)} < 1$$

where j is the Maximal Gap ordinal number. The ratio at $j = 18$, $p_i = 1357201$ attains approx. 1.01717452 and strictly increases at every step j , diverging as j

increases. Please refer to Figure 5. Consequently this implies, that we have a contradiction to the initial assumption. This implies that:

$$(2.24) \quad \lim_{p_i \rightarrow \infty} \left(C_1 \left(\frac{(\log_{10} p_i)^\alpha}{(\sqrt{5} - 1)^\beta} \right) \left[\sum_{k=0}^{(lmt-1)} \left(\frac{1}{(\log(x_1^* + k\Delta x))^\beta} \right) \Delta x \right] - \mathfrak{g} \right) \rightarrow \infty$$

which in turn implies that Theorem 2.7 holds for $p_i \in \mathbb{N} \mid p_i \geq 1357201$. For $155921 \leq p_i \leq 1357201$ direct computer evaluation demonstrates that, Theorem 2.7 holds in this range as well. Please refer to Table 3. Hence, Theorem 2.7 holds as stated for all $p_i \in \mathbb{N} \mid p_i \geq 155921$. Concluding the proof. $\square$

FIGURE 5. The drawing shows ratio 2.23. The figure is drawn w.r.t. $p_i \in \mathbb{N}$, s.t. $1357201 \leq p_n \leq 18361375334787046697$.

3. LOWER BOUND ON MAXIMAL PRIME GAPS WITH HISTORICAL BACKGROUND

From the Prime Number Theorem we have that an average gap between consecutive primes is given by $\log n$ for any $n \in \mathbb{N}$. There exist however prime gaps much shorter - containing only a single composite number, and gaps which are much longer than average - the maximal prime gaps. In 1929 R. Backlund [1] published a paper in which he proved the lower bound on the maximal prime gaps:

$$(3.1) \quad \mathfrak{g} = p_{(n+1)} - p_{(n)} > (2 - \epsilon) \log p_{(n)} \text{ for any } \epsilon > 0$$

This was the first notable result in this area. In 1931 Westzynthius [31] proved the normalized prime gap limit superior:

$$(3.2) \quad \limsup_{p_{(n+1)} \rightarrow \infty} \left(\frac{\mathfrak{g}}{\log p_{(n+1)}} \right) = \limsup_{p_{(n+1)} \rightarrow \infty} \left(\frac{p_{(n+1)} - p_{(n)}}{\log p_{(n+1)}} \right) \rightarrow \infty$$

The result 3.1 had been improved upon in 1935 by Paul Erdős [10] who proved that for $c > 0$ we have:

$$(3.3) \quad \mathfrak{g} = p_{(n+1)} - p_{(n)} > \frac{c(\log N)(\log \log N)}{(\log \log \log N)^2}$$

Robert Rankin [24] in 1938 proved that for $c > 0$ we have:

$$(3.4) \quad \mathfrak{g} = p_{(n+1)} - p_{(n)} > \frac{c(\log N)(\log \log N)(\log \log \log N)}{(\log \log \log N)^2}$$

In a more recent development, Kevin Ford Et Al. [16] in 2016 established for all $p_{(n+1)} \in \mathbb{N} \mid p_{(n+1)} \leq N$, a bound of the form:

$$(3.5) \quad \mathfrak{g} = p_{(n+1)} - p_{(n)} \gg \frac{(\log N)(\log \log N)(\log \log \log \log N)}{\log \log \log N}$$

with the constant c being implicit. For clarity of presentation, some graphs implement a variant of a logarithmic scaling of the horizontal axis, given by:

Definition 3.1 (Scaling factor). $\xi = \frac{\log_{10}(\frac{p_{(n)}}{24})}{\log_{10}(24)}$

Definition 3.2 (Alpha Constant). $\alpha = \left(\frac{(5 + (\log 10)^2)}{(\log 10)^2} \right)$

Definition 3.3 (Beta Constant). $\beta = \left(\frac{(\exp 1)\pi}{2} \right)$

Theorem 3.4 (Maximal Prime Gaps Lower Bound).

The maximal prime gaps Lower Bound, for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 23$ is given by:

$$(3.6) \quad \mathfrak{g} = p_{(n+1)} - p_{(n)} \geq \mathcal{LB}_{p_n} = \mathcal{M} \times \left(5 (\log_{10} p_n)^\alpha - 7 (\log_{10} p_n)^{(2\gamma)} \right) + \beta$$

Where α and β are given by the Definitions 3.2 and 3.3 respectively, γ is the Euler-Mascheroni constant and $\mathcal{M}$ is given by:

$$(3.7) \quad \mathcal{M} = \left(1 - \frac{\sqrt{5} - 1}{\log_{10} p_n} \right).$$

Proof.

Theorem 1.1 gives for all $p_n \in \mathbb{N} \mid p_n \geq 11$, the upper bound on the maximal prime gaps:

$$(3.8) \quad \mathfrak{g} = (p_{n+1} - p_n) \leq 5 (\log_{10} p_n)^2$$

Clearly, for all $p_n \in \mathbb{N} \mid p_n \geq 23$:

$$(3.9) \quad \mathcal{LB}_{p_n} = \mathcal{M} \times \left(5 (\log_{10} p_n)^\alpha - 7 (\log_{10} p_n)^{(2\gamma)} \right) + \beta < 5 (\log_{10} p_n)^2$$

Suppose that Theorem 3.4 is false, for $p_n \in \mathbb{N} \mid p_n \geq 155921$. Consequently,

$$(3.10) \quad \mathcal{LB}_{p_n} = \mathcal{M} \times \left(5 (\log_{10} p_n)^\alpha - 7 (\log_{10} p_n)^{(2\gamma)} \right) + \beta > (p_{n+1} - p_n) = \mathfrak{g}$$

This implies that:

$$(3.11) \quad \frac{p_n}{\mathcal{M} \times \left(5 (\log_{10} p_n)^\alpha - 7 (\log_{10} p_n)^{(2\gamma)} \right) + \beta} - \frac{p_n}{(p_{n+1} - p_n)} < 0$$

Cauchy Root Test at $j = 13$, $p_n = 155921$:

(3.12)

$$\sqrt[j]{|a_j|} = \sqrt[j]{\left| \left(\frac{p_n}{\mathcal{M} \times \left(5 (\log_{10} p_n)^\alpha - 7 (\log_{10} p_n)^{(2\gamma)} \right) + \beta} \right) - \left(\frac{p_n}{\mathfrak{g}} \right) \right|} \approx 1.72585$$

Where j is the Ordinal Number of the maximal gap and p_n is the prime at which the maximal gap commences. Furthermore, the Cauchy's Root Test at every step j exceeds 1, while decreasing strictly from above towards 1. Please refer to Figure 6. Hence, by the definition of the Root Test, the sequence $|a_j|$ diverges. In accordance with the initial hypothesis, this implies that:

$$(3.13) \quad \left(\frac{p_n}{\mathcal{M} \times \left(5 (\log_{10} p_n)^\alpha - 7 (\log_{10} p_n)^{(2\gamma)} \right) + \beta} - \frac{p_n}{(p_{n+1} - p_n)} \right) \rightarrow -\infty$$

However, at $p_n = 155921$ the difference 3.13 attains approx. 698.282 and rapidly diverges. Please refer to Figure 6. Hence, we have a contradiction to the initial assumption, necessarily this implies that for any $p_n \geq 155921$:

$$(3.14) \quad \mathcal{M} \times \left(5 (\log_{10} p_n)^\alpha - 7 (\log_{10} p_n)^{(2\gamma)} \right) + \beta < \mathfrak{g}$$

Hence, Theorem 3.4 holds for $p_n \in \mathbb{N} \mid p_n \geq 155921$. For $23 \leq p_n \leq 155921$, direct computer evaluation demonstrates, that Theorem 3.4 holds in this range as well. Please refer to Table 4. This implies that Theorem 3.4 holds as stated for all $p_n \in \mathbb{N} \mid p_n \geq 23$. Concluding the proof. $\square$

FIGURE 6. The L.H. log-log drawing shows the graph of the difference of ratios 3.11. The R.H. log-linear drawing shows the graph of the Cauchy Root Test 3.12. The figures are drawn w.r.t. $p_n \in \mathbb{N}$, in the range $23 \leq p_n \leq 18361375334787046697$.

FIGURE 7. These log-log drawings show the graphical comparison of the two maximal Prime Gaps Lower Bounds: Ford et al $\mathcal{PB}$ (Blue) and the $\mathcal{LB}$ (Red). The R.H. drawing shows the difference $\mathcal{LB} - \mathcal{PB}$. Both figures are drawn over the range $p_{(n+1)} \in \mathbb{N} \mid 4652507 \leq p_{(n+1)} \leq 18361375334787048247$.

The lower bound on Maximal Prime Gaps, presented by Ford et al [16], attains values that are less than zero over the range of the maximal prime gaps for $p_{(n+1)} \in \mathbb{N} \mid 11 \leq p_{(n+1)} \leq 2010881$. Please refer to Table 5 in the Appendix. Consequently, the log-log figures can only be drawn for $p_{(n+1)} \in \mathbb{N} \mid p_{(n+1)} \geq 4652507$. Pragmatically, we prove Theorem 3.5, beginning at the lowest $\mathcal{C}$ value. This means, at the prime gap terminating at $p_{(n+1)} = 19581334193189$. **The value of the constant $\mathcal{C}$ is only an approximation, as it is computed w.r.t. the Lower Bound, given by Theorem 3.4.** The approximation is however sufficient and suitable, to compute the approximate value of $\mathcal{C}$, at any technically feasible (or desired) point $n \in \mathbb{N} \mid n \geq 4652507$.

Corollary 3.5 (Maximal Prime Gaps Lower Bound $\mathcal{C}$ Value).

The primary lower bound for all $p_{(n+1)} \in \mathbb{N} \mid p_{(n+1)} \geq 4652507$ (Ford et al [16]), is given by:

$$(3.15) \quad \mathfrak{g} = p_{(n+1)} - p_{(n)} \gg \frac{(\log N) (\log \log N) (\log \log \log \log N)}{\log \log \log N} = \mathcal{PB}$$

The implicit constant $\mathcal{C}$ can be estimated by implementing $\mathcal{C} \gtrsim \mathcal{LB}_{p_{(n+1)}} / \mathcal{PB}_{p_{(n+1)}}$. The value of $\mathcal{C}$, for $p_{(n+1)} \geq 4652507$, is $\mathcal{C} \geq 32$, and gradually increases unboundedly as $p_{(n+1)}$ tends to infinity.

Remark 3.1. Corollary 3.5 is a direct consequence of Theorem 3.4. Hence, a simple validation is sufficient.

Proof.

By Theorems 1.1 and 3.4 we have:

$$(3.16) \quad \left(\mathcal{M} \times \left(5 (\log_{10} p_n)^\alpha - 7 (\log_{10} p_n)^{(2\gamma)} \right) + \beta \right) \leq (p_{n+1} - p_n) \leq 5 (\log_{10} p_n)^2$$

Kevin Ford Et Al. [16] have shown, that for any sufficiently large $p_{(n+1)} \leq N$:

$$(3.17) \quad \mathfrak{g} = p_{(n+1)} - p_{(n)} \gg \frac{(\log p_{(n+1)}) (\log \log p_{(n+1)}) (\log \log \log \log p_{(n+1)})}{\log \log \log p_{(n+1)}} = \mathcal{PB}$$

The difference:

$$(3.18) \quad (\mathcal{LB}_{p_{(n+1)}} - \mathcal{PB}_{p_{(n+1)}})$$

At $p_{(n+1)} = 19581334193189$ attains 552.277785 and increases. Please refer to Figure 7 and Table 5. Implementing Cauchy Root Test:

$$(3.19) \quad \sqrt[j]{\mathcal{C}_j} = \sqrt[j]{\left(\frac{\mathcal{LB}_{p_{(n+1)}}}{\mathcal{PB}_{p_{(n+1)}}}\right)}$$

where $j \in \mathbb{N} \mid j \geq 55$ is the ordinal number of the maximal prime gap. At $p_{(n+1)} = 19581334193189$ the Cauchy Root Test attains approx. 1.06525 and converges asymptotically, strictly from above to 1. Please refer to Table 5. Consequently, by the definition of the Cauchy Root Test, the ratio $\mathcal{C}_j$ diverges for $p_{(n+1)} \geq 19581334193189$. This implies that the difference 3.18 also diverges. The ratio $\mathcal{C}_j = (\mathcal{LB}_{p_{(n+1)}}) / (\mathcal{PB}_{p_{(n+1)}})$ at $p_{(n+1)} = 19581334193189$, attains 32.3403523719 and further increases at every step j , as $p_{(n+1)}$ increases unboundedly. Please refer to Figure 8. This shows that Corollary 3.5 is valid for $p_{(n+1)} \geq 19581334193189$. For $4652507 \leq p_{(n+1)} \leq 19581334193189$, Figure 8 shows that Corollary 3.5 is true in that range too. Thus, completing the validation. $\square$

FIGURE 8. These log-log drawings show the ratio $\frac{\mathcal{LB}}{\mathcal{PB}}$.

The L.H. figure is drawn over the range:

$$p_{(n+1)} \in \mathbb{N} \mid 4652507 \leq p_{(n+1)} \leq 18361375334787048247.$$

The R.H. figure is drawn over the range:

$$p_{(n+1)} \in \mathbb{N} \mid 10726905041 \leq p_{(n+1)} \leq 18361375334787048247.$$

The global minimum occurs at $p_{(n+1)} = 19581334193189$ where the ratio attains approx. 32.34035237.

The graphs in the Figure 9 have been generated to simply assist in visualization, of the $\mathcal{C}$ constant's slow rate of divergence and/or convergence of the Cauchy Root Test. Although the figures are not a part of the verification itself, they aid to understand the concept and help to shape up our expectations.

FIGURE 9. The L.H. drawing shows the Cauchy Root Test of the ratio $\frac{\mathcal{LB}}{\mathcal{PB}}$. The R.H. drawing shows the ratio $\frac{\mathcal{LB}}{\mathcal{PB}}$. Both figures are drawn in the range $n \in \mathbb{N} \mid 4652507 \leq \approx 7.662607 \times 10^{1499}$.

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4. APPENDIX

4.1. Tabular data.

TABLE 1

Maximal Prime Gaps Upper Bound			
N°	Gap start p_n	Actual gap g	$\mathcal{UB}_{(p_n)}$
1	7	4	3
2	23	6	9
3	89	8	19
4	113	14	21
5	523	18	36
6	887	20	43
7	1129	22	46
8	1327	34	48
9	9551	36	79
10	15683	44	88
11	19609	52	92
12	31397	72	101
13	155921	86	134
14	360653	96	154
15	370261	112	155
16	492113	114	161
17	1349533	118	187
18	1357201	132	188
19	2010733	148	198
20	4652353	154	222
21	17051707	180	261
22	20831323	210	267
23	47326693	220	294
24	122164747	222	326
25	189695659	234	342
26	191912783	248	343
27	387096133	250	368
28	436273009	282	373
29	1294268491	288	415
30	1453168141	292	419
31	2300942549	320	438
32	3842610773	336	459
33	4302407359	354	464
34	10726904659	382	503
35	20678048297	384	532
36	22367084959	394	535
37	25056082087	456	540
38	42652618343	464	564
39	127976334671	468	616
Continued ...			

TABLE 1. Continued

Maximal Prime Gaps Upper Bound (Continued)			
40	182226896239	474	634
41	241160624143	486	647
42	297501075799	490	658
43	303371455241	500	659
44	304599508537	514	659
45	416608695821	516	675
46	461690510011	532	680
47	614487453523	534	694
48	738832927927	540	704
49	1346294310749	582	735
50	1408695493609	588	737
51	1968188556461	602	755
52	2614941710599	652	770
53	7177162611713	674	826
54	13829048559701	716	863
55	19581334192423	766	883
56	42842283925351	778	929
57	90874329411493	804	974
58	171231342420521	806	1012
59	218209405436543	906	1028
60	1189459969825483	916	1136
61	1686994940955803	924	1159
62	1693182318746371	1132	1159
63	43841547845541059	1184	1384
64	55350776431903243	1198	1401
65	80873624627234849	1220	1429
66	203986478517455989	1224	1498
67	218034721194214273	1248	1503
68	305405826521087869	1272	1528
69	352521223451364323	1328	1539
70	401429925999153707	1356	1549
71	418032645936712127	1370	1552
72	804212830686677669	1442	1603
73	1425172824437699411	1476	1647
74	5733241593241196731	1488	1759
75	6787988999657777797	1510	1773
76	15570628755536096243	1526	1841
77	17678654157568189057	1530	1852
78	18361375334787046697	1550	1855
The End			

TABLE 2

Infimum on the Maximal Prime Gaps				
N°	Gap start p_n	Actual gap g	Infimum	Est. Error
1	7	4	—	—
2	23	6	5.78012	0.21988
3	89	8	7.922911	0.077089
4	113	14	13.826016	0.173984
5	523	18	17.950855	0.049145
6	887	20	19.966698	0.033302
7	1129	22	21.969323	0.030677
8	1327	34	33.939380	0.06062
9	9551	36	35.991954	0.008046
10	15683	44	43.992819	0.007181
11	19609	52	51.992085	0.007915
12	31397	72	71.99072	0.00928
13	155921	86	85.996433	0.003567
14	360653	96	95.997221	0.002779
15	370261	112	111.996601	0.003399
16	492113	114	113.996872	0.003128
17	1349533	118	117.997421	0.002579
18	1357201	132	131.997069	0.002931
19	2010733	148	147.996839	0.003161
20	4652353	154	153.996917	0.003083
21	17051707	180	179.9965	0.0035
22	20831323	210	209.995915	0.004085
23	47326693	220	219.995734	0.004266
24	122164747	222	221.995687	0.004313
25	189695659	234	233.995446	0.004554
26	191912783	248	247.995172	0.004828
27	387096133	250	249.995118	0.004882
28	436273009	282	281.994489	0.005511
29	1294268491	288	287.99434	0.00566
30	1453168141	292	291.994258	0.005742
31	2300942549	320	319.993692	0.006308
32	3842610773	336	335.993359	0.006641
33	4302407359	354	353.9929995	0.0070005
34	10726904659	382	381.992411	0.007589
35	20678048297	384	383.992347	0.007653
36	22367084959	394	393.992145	0.007855
37	25056082087	456	455.990905	0.009095
38	42652618343	464	463.990722	0.009278
39	127976334671	468	467.990596	0.009404
40	182226896239	474	473.99046	0.00954
41	241160624143	486	485.990207	0.009793
42	297501075799	490	489.990118	0.009882
Continued ...				

TABLE 2. Continued

Infimum (Continued)				
43	303371455241	500	499.989915	0.010085
44	304599508537	514	513.989632	0.010368
45	416608695821	516	515.989578	0.010422
46	461690510011	532	531.989251	0.010749
47	614487453523	534	533.989198	0.010802
48	738832927927	540	539.989067	0.010933
49	1346294310749	582	581.988189	0.011811
50	1408695493609	588	587.988065	0.011935
51	1968188556461	602	601.987765	0.012235
52	2614941710599	652	651.986733	0.013267
53	7177162611713	674	673.986233	0.013767
54	13829048559701	716	715.98534	0.01466
55	19581334192423	766	765.984297	0.015703
56	42842283925351	778	777.984006	0.015994
57	90874329411493	804	803.983429	0.016571
58	171231342420521	806	805.983352	0.016648
59	218209405436543	906	905.981272	0.018728
60	1189459969825483	916	915.98096	0.01904
61	1686994940955803	924	923.980774	0.019226
62	1693182318746371	1132	1131.976444	0.023556
63	43841547845541059	1184	1183.975126	0.024874
64	55350776431903243	1198	1197.974816	0.025184
65	80873624627234849	1220	1219.974325	0.025675
66	203986478517455989	1224	1223.974178	0.025822
67	218034721194214273	1248	1247.973664	0.026336
68	305405826521087869	1272	1271.973136	0.026864
69	352521223451364323	1328	1327.971942	0.028058
70	401429925999153707	1356	1355.97134	0.02866
71	418032645936712127	1370	1369.971039	0.028961
72	804212830686677669	1442	1441.969466	0.030534
73	1425172824437699411	1476	1475.9687	0.0313
74	5733241593241196731	1488	1487.968334	0.031666
75	6787988999657777797	1510	1509.967856	0.032144
76	15570628755536096243	1526	1525.967448	0.032552
77	17678654157568189057	1530	1529.967353	0.032647
78	18361375334787046697	1550	1549.966923	0.033077
The End				

TABLE 3

Supremum on the Maximal Prime Gaps				
N°	Gap start p_n	Actual gap g	Supremum	Est. Error
1	7	4	—	—
2	23	6	5.780362	-0.219638
3	89	8	7.923243	-0.076757
4	113	14	13.826595	-0.173405
5	523	18	17.951607	-0.048393
6	887	20	19.967535	-0.032465
7	1129	22	21.970244	-0.029756
8	1327	34	33.940803	-0.059197
9	9551	36	35.993464	-0.006536
10	15683	44	43.994664	-0.005336
11	19609	52	51.994266	-0.005734
12	31397	72	71.993739	-0.006261
13	155921	86	86.00004	0.00004
14	360653	96	96.001247	0.001247
15	370261	112	112.001298	0.001298
16	492113	114	114.001654	0.001654
17	1349533	118	118.002371	0.002371
18	1357201	132	132.002606	0.002606
19	2010733	148	148.003048	0.003048
20	4652353	154	154.003377	0.003377
21	17051707	180	180.004052	0.004052
22	20831323	210	210.004725	0.004725
23	47326693	220	220.004964	0.004964
24	122164747	222	222.005001	0.005001
25	189695659	234	234.005263	0.005263
26	191912783	248	248.005578	0.005578
27	387096133	250	250.005607	0.005607
28	436273009	282	282.006321	0.006321
29	1294268491	288	288.006424	0.006424
30	1453168141	292	292.006510	0.006510
31	2300942549	320	320.007119	0.007119
32	3842610773	336	336.007458	0.007458
33	4302407359	354	354.007854	0.007854
34	10726904659	382	382.008441	0.008441
35	20678048297	384	384.008461	0.008461
36	22367084959	394	394.008679	0.008679
37	25056082087	456	456.01004	0.01004
38	42652618343	464	464.010193	0.010193
39	127976334671	468	468.010236	0.010236
40	182226896239	474	474.010352	0.010352
41	241160624143	486	486.010603	0.010603
42	297501075799	490	490.010681	0.010681
Continued ...				

TABLE 3. Continued

Supremum (Continued)				
43	303371455241	500	500.010898	0.010898
44	304599508537	514	514.011203	0.011203
45	416608695821	516	516.011233	0.011233
46	461690510011	532	532.011577	0.011577
47	614487453523	534	534.011608	0.011608
48	738832927927	540	540.011730	0.011730
49	1346294310749	582	582.012614	0.012614
50	1408695493609	588	588.012742	0.012742
51	1968188556461	602	602.013029	0.013029
52	2614941710599	652	652.014097	0.014097
53	7177162611713	674	674.014521	0.014521
54	13829048559701	716	716.015391	0.015391
55	19581334192423	766	766.016446	0.016446
56	42842283925351	778	778.016659	0.016659
57	90874329411493	804	804.017175	0.017175
58	171231342420521	806	806.017181	0.017181
59	218209405436543	906	906.019298	0.019298
60	1189459969825483	916	916.019409	0.019409
61	1686994940955803	924	924.019559	0.019559
62	1693182318746371	1132	1132.02396	0.02396
63	43841547845541059	1184	1184.024828	0.024828
64	55350776431903243	1198	1198.025106	0.025106
65	80873624627234849	1220	1220.025540	0.025540
66	203986478517455989	1224	1224.025558	0.025558
67	218034721194214273	1248	1248.026056	0.026056
68	305405826521087869	1272	1272.026532	0.026532
69	352521223451364323	1328	1328.02769	0.02769
70	401429925999153707	1356	1356.028262	0.028262
71	418032645936712127	1370	1370.028553	0.028553
72	804212830686677669	1442	1442.030	0.030
73	1425172824437699411	1476	1476.030664	0.030664
74	5733241593241196731	1488	1488.030804	0.030804
75	6787988999657777797	1510	1510.031245	0.031245
76	15570628755536096243	1526	1526.031511	0.031511
77	17678654157568189057	1530	1530.031588	0.031588
78	18361375334787046697	1550	1550.031998	0.031998
The End				

TABLE 4

Maximal Prime Gaps Lower Bound			
N°	Gap start p_n	Actual gap g	$\mathcal{LB}_{(p_n)}$
1	7	4	—
2	23	6	4.187953
3	89	8	5.42782
4	113	14	5.929179
5	523	18	11.194365
6	887	20	13.835209
7	1129	22	15.182401
8	1327	34	16.134312
9	9551	36	30.970394
10	15683	44	35.630568
11	19609	52	37.852172
12	31397	72	42.781476
13	155921	86	62.087339
14	360653	96	73.741040
15	370261	112	74.123637
16	492113	114	78.330546
17	1349533	118	94.233928
18	1357201	132	94.327589
19	2010733	148	100.943659
20	4652353	154	115.842369
21	17051707	180	141.000844
22	20831323	210	145.104297
23	47326693	220	162.552277
24	122164747	222	183.972161
25	189695659	234	194.369152
26	191912783	248	194.647620
27	387096133	250	211.835859
28	436273009	282	214.838993
29	1294268491	288	243.124468
30	1453168141	292	246.240438
31	2300942549	320	258.803570
32	3842610773	336	273.193759
33	4302407359	354	276.417801
34	10726904659	382	303.175118
35	20678048297	384	323.162466
36	22367084959	394	325.596419
37	25056082087	456	329.131748
38	42652618343	464	345.952387
39	127976334671	468	382.020819
40	182226896239	474	394.001476
41	241160624143	486	403.631907
42	297501075799	490	410.923740
Continued ...			

TABLE 4. Continued

Maximal Prime Gaps Lower Bound (Continued)			
43	303371455241	500	411.60569
44	304599508537	514	411.746751
45	416608695821	516	422.754414
46	461690510011	532	426.397665
47	614487453523	534	436.616941
48	738832927927	540	443.268043
49	1346294310749	582	465.271350
50	1408695493609	588	466.954333
51	1968188556461	602	479.471048
52	2614941710599	652	490.233868
53	7177162611713	674	529.440803
54	13829048559701	716	555.711929
55	19581334192423	766	569.899725
56	42842283925351	778	602.487386
57	90874329411493	804	634.630708
58	171231342420521	806	662.354842
59	218209405436543	906	673.119558
60	1189459969825483	916	750.817572
61	1686994940955803	924	767.350344
62	1693182318746371	1132	767.524491
63	43841547845541059	1184	930.030304
64	55350776431903243	1198	942.263033
65	80873624627234849	1220	962.330541
66	203986478517455989	1224	1012.166929
67	218034721194214273	1248	1015.802441
68	305405826521087869	1272	1034.296059
69	352521223451364323	1328	1042.219511
70	401429925999153707	1356	1049.420519
71	418032645936712127	1370	1051.671730
72	804212830686677669	1442	1088.346960
73	1425172824437699411	1476	1120.927172
74	5733241593241196731	1488	1202.164233
75	6787988999657777797	1510	1212.210242
76	15570628755536096243	1526	1262.198192
77	17678654157568189057	1530	1269.930815
78	18361375334787046697	1550	1272.242921
The End			

TABLE 5

Maximal Prime Gaps Lower Bound (Ford et al).				
N°	Max. Prime Gap Endp.	$\mathcal{PB}_{(p_{n+1})}$	Diff: $\mathcal{LB} - \mathcal{PB}$	Cauchy Root Test
1	11	—	—	—
2	29	-34.555205	38.764478	—
3	97	-14.436398	20.035106	—
4	127	-13.160648	19.366295	—
5	541	-9.404357	20.755163	—
6	907	-8.592431	22.548443	—
7	1151	-8.258842	23.552841	—
8	1361	-8.035861	24.322807	—
9	9587	-5.838985	36.843322	—
10	15727	-5.342327	41.000285	—
11	19661	-5.121875	43.000837	—
12	31469	-4.662480	47.468766	—
13	156007	-3.121775	65.216427	—
14	360749	-2.314771	76.059679	—
15	370373	-2.289334	76.417379	—
16	492227	-2.014078	80.348099	—
17	1349651	-1.029834	95.265207	—
18	1357333	-1.024255	95.353452	—
19	2010881	-0.636680	101.58160	—
20	4652507	0.1990	115.643978	1.37484
21	17051887	1.517748	139.483311	1.24084
22	20831533	1.723817	143.380688	1.22323
23	47326913	2.576437	159.975942	1.19746
24	122164969	3.577814	180.394389	1.17841
25	189695893	4.048367	190.320814	1.16749
26	191913031	4.060843	190.586809	1.16049
27	387096383	4.818926	207.016949	1.15041
28	436273291	4.949078	209.889930	1.14416
29	1294268779	6.144875	236.979598	1.13522
30	1453168433	6.273519	239.966925	1.13013
31	2300942869	6.786506	252.017068	1.12463
32	3842611109	7.363520	265.830242	1.11955
33	4302407713	7.491331	268.926473	1.11554
34	10726905041	8.532880	294.642239	1.11072
35	20678048681	9.290264	313.872203	1.10673
36	22367085353	9.381377	316.215042	1.10354
37	25056082543	9.513303	319.618446	1.10051
38	42652618807	10.134455	335.817933	1.09736
39	127976335139	11.432607	370.588212	1.09415
40	182226896713	11.854422	382.147054	1.09154
41	241160624629	12.190328	391.441579	1.08911
42	297501076289	12.442850	398.48089	1.08683
Continued ...				

TABLE 5. Continued

Maximal Prime Gaps Bound Ford et al (Continued)				
43	303371455741	12.466388	399.139302	1.08472
44	304599509051	12.471255	399.275495	1.08272
45	416608696337	12.849336	409.905078	1.08073
46	461690510543	12.973730	413.423935	1.07888
47	614487454057	13.320741	423.2962	1.07708
48	738832928467	13.545106	429.722937	1.07538
49	1346294311331	14.279345	450.992005	1.07369
50	1408695494197	14.335014	452.619319	1.07215
51	1968188557063	14.746922	464.724126	1.07065
52	2614941711251	15.09820	475.135668	1.06922
53	7177162612387	16.356341	513.084462	1.06781
54	13829048560417	17.181710	538.530219	1.0665
55	19581334193189	17.621939	552.277785	1.06525
56	42842283926129	18.619276	583.86811	1.06405
57	90874329412297	19.585240	615.045467	1.06292
58	171231342421327	20.405138	641.949704	1.06184
59	218209405437449	20.720335	652.399224	1.06077
60	1189459969826399	22.946864	727.870708	1.05986
61	1686994940956727	23.410328	743.940015	1.05888
62	1693182318747503	23.415192	744.109299	1.0579
63	43841547845542243	27.803846	902.226458	1.0573
64	55350776431904441	28.123132	914.139901	1.0564
65	80873624627236069	28.643866	933.686676	1.05556
66	203986478517457213	29.921297	982.245633	1.0548
67	218034721194215521	30.013632	985.788809	1.05397
68	305405826521089141	30.481598	1003.814461	1.0532
69	352521223451365651	30.681217	1011.538294	1.05242
70	401429925999155063	30.862185	1018.558334	1.05167
71	418032645936713497	30.918672	1020.753057	1.05093
72	804212830686679111	31.833185	1056.513775	1.05028
73	1425172824437700887	32.636763	1088.290409	1.04964
74	5733241593241198219	34.606352	1167.557881	1.04911
75	6787988999657779307	34.846692	1177.363551	1.04846
76	15570628755536097769	36.032573	1226.165619	1.0479
77	17678654157568190587	36.214561	1233.716254	1.04728
78	18361375334787048247	36.268902	1235.974019	1.04667
The End				