Model order reduction of dynamic skeletal muscle models

Mylena Mordhorst, Daniel Wirtz, Oliver Röhrle

Institute of Applied Mechanics (CE) & Stuttgart Research Centre for Simulation Technology (SimTech), University of Stuttgart, 70569 Stuttgart, Germany

Motivation

Forward simulations of three-dimensional continuum-mechanical skeletal muscle models are a complex and computationally expensive problem. Moreover, considering a fully dynamic modelling framework based on the theory of finite elasticity is challenging as the muscles' mechanical behaviour requires to consider a highly nonlinear, anisotropic, viscoelastic and incompressible material behaviour. The governing equations yield a nonlinear second-order differential algebraic equation (DAE) system, which represents a challenge for model order reduction techniques.

Governing equations

Let $\Omega_0 \subset \mathbb{R}^3$ be the muscle domain in the reference configuration. The equations describing the muscle deformation over time are the balance of momentum subject to the incompressibility constraint

$$\begin{aligned} \rho_0(X)\frac{\partial \boldsymbol{V}}{\partial t}(X,t) &= \nabla \boldsymbol{P}(X,t) + \boldsymbol{B}(X,t)\,, \quad \forall X \in \Omega_0\,, \\ J(X,t) &= 1\,, \quad \forall X \in \Omega_0\,, \end{aligned} \tag{1}$$

and a nonlinear constitutive equation of the form

$$\begin{aligned} \boldsymbol{P}(X,t) =& \boldsymbol{P}^{iso}(X,t) + \boldsymbol{P}^{aniso}(X,t) + \boldsymbol{P}^{act}(X,t) + \boldsymbol{P}^{visc}(X,t) \\ &+ p(X,t)\boldsymbol{F}^{-T}(X,t)\,. \end{aligned} \tag{2}$$

Herein ρ_0 is the muscle density in the reference configuration, $\boldsymbol{V}$ is the velocity field, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, $\boldsymbol{B}$ are the body forces, $J := \det \boldsymbol{F}$ is the Jacobian, $\boldsymbol{F}$ is the deformation gradient and p is the hydrostatic pressure.

Inserting equation (2) into (1) and discretising the resulting equation using the finite element method (Taylor-Hood elements), one obtains the following second-order DAE system

$$\begin{aligned} \boldsymbol{M}\boldsymbol{u}''(t) + \boldsymbol{D}\boldsymbol{u}'(t) + \boldsymbol{K}(\boldsymbol{u}(t), \boldsymbol{w}(t)) &= \boldsymbol{0}\,, \\ \text{s.t. } \boldsymbol{g}(\boldsymbol{u}(t)) &= \boldsymbol{0}\,, \end{aligned} \tag{3}$$

where $\boldsymbol{u}(t) \in \mathbb{R}^d$ is the vector of position coefficients, $\boldsymbol{w}(t) \in \mathbb{R}^{d_p}$ contains the pressure coefficients and $\boldsymbol{M} \in \mathbb{R}^{d\times d}$ and $\boldsymbol{D} \in \mathbb{R}^{d\times d}$ are the mass and viscous damping matrix, respectively. Further, $\boldsymbol{K} : \mathbb{R}^{d\times d_p} \to \mathbb{R}^d$ is the generalised stiffness operator for the nonlinear parts of $\boldsymbol{P}$, and $\boldsymbol{g} : \mathbb{R}^d \to \mathbb{R}^{d_p}$ is the operator associated with the incompressibility constraint.

Methods

So far, the full system (3) in matrix format, i.e.

$$\begin{pmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \end{pmatrix}'' + \begin{pmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \end{pmatrix}' + \begin{pmatrix} \boldsymbol{K}(\boldsymbol{u},\boldsymbol{w}) \\ \boldsymbol{g}(\boldsymbol{u}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}\,, \tag{4}$$

is solved by transforming it into a first-order system introducing the velocity $\boldsymbol{v}(t) := \boldsymbol{u}'(t)$ and by applying a numerical time integration of BDF-type (linear multistep method, MATLAB solver ode15i).

To obtain the reduced system, we apply a Galerkin block-projection of the second-order system (4), where the algebraic DOFs $\boldsymbol{w}$ are not reduced, i.e. we set $\boldsymbol{u} \approx \boldsymbol{V}\boldsymbol{z}_1$, with $\boldsymbol{V} \in \mathbb{R}^{d\times r}$, $\boldsymbol{z}_1 \in \mathbb{R}^r$ and $r \ll d$. Only afterwards, for the solution process, the reduced system is transformed into a first-order system by setting $\boldsymbol{z}_2 := \boldsymbol{z}_1'$, yielding the parametric, nonlinear, dynamical, first-order system

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}^T\boldsymbol{M}\boldsymbol{V} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{w} \end{pmatrix}' + \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}^T\boldsymbol{D}\boldsymbol{V} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{w} \end{pmatrix} + \begin{pmatrix} -\boldsymbol{z}_2 \\ \boldsymbol{V}^T\boldsymbol{K}(\boldsymbol{V}\boldsymbol{z}_1, \boldsymbol{w}) \\ \boldsymbol{g}(\boldsymbol{V}\boldsymbol{z}_1) \end{pmatrix} = \boldsymbol{0}\,.$$

While the reduced basis $\boldsymbol{V}$ is computed in an offline phase based on precomputed trajectories via the POD-Greedy algorithm, the evaluation of the projected nonlinear term can be pursued by a DEIM approximation scheme, i.e. $\boldsymbol{V}^T\boldsymbol{K}(\boldsymbol{V}\boldsymbol{z}_1, \boldsymbol{w}) \approx \boldsymbol{V}^T\boldsymbol{U}(\boldsymbol{P}^T\boldsymbol{U})^{-1}\boldsymbol{P}^T\boldsymbol{K}(\boldsymbol{V}\boldsymbol{z}_1, \boldsymbol{w})$ with DEIM basis and DEIM points $\boldsymbol{U}, \boldsymbol{P} \in \mathbb{R}^{d\times m}$ and $m \ll d$ being the DEIM order.

Results

To test the proposed method, we perform the following experiment with a fusiform muscle, which contains fibres in y-direction and which is fixed on its left end. The muscle is discretised using 96 elements, which corresponds to a total of 7521 DOFs. We consider three modelling parameters μ_1, μ_2, μ_3. During a simulation of 200 ms, the right end is subject to a traction force μ_1, which is applied linearly from 0-30 ms. Then, starting at 50 ms the muscle is fully activated within μ_2 ms. The third parameter, μ_3, is the viscosity of the muscle tissue.

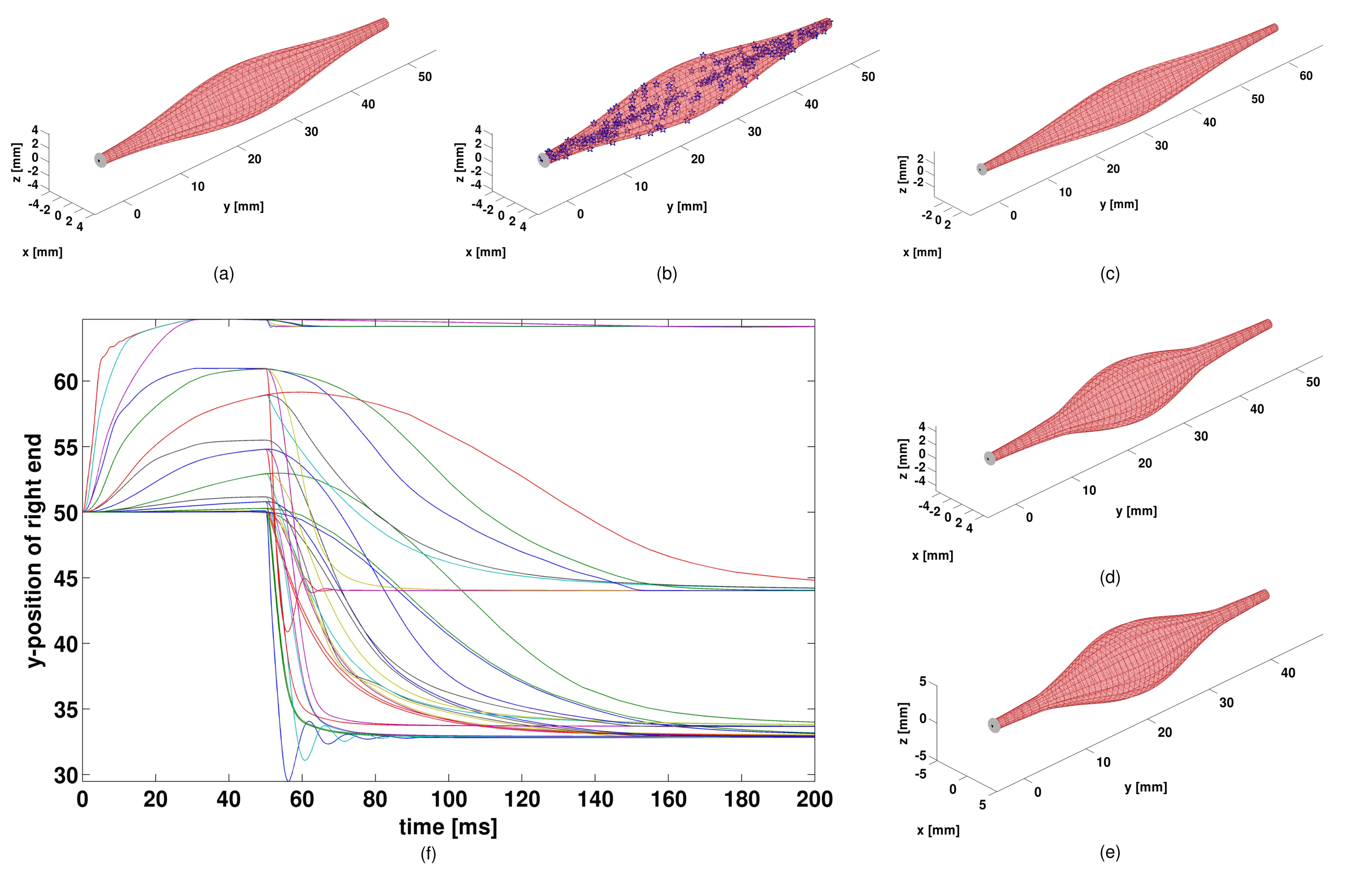

Figure: The dynamics of the full system: (a) undeformed muscle geometry, (b) chosen DEIM points, (c)-(e) the deformation at 50 ms, 100 ms and 200 ms respectively, and (f) the mean y-position of the right end over time for the 45 precomputed trajectories.

During the offline phase, 45 trajectories were computed for different parameter combinations ($\mu_1 \in \{0.001, 0.01, 0.1, 1, 10\}$MPa, $\mu_2 \in \{1, 10, 100\}$ ms, $\mu_3 \in \{0.5, 2.25, 10\}$ kP). The computation of the trajectory data took $\sim$ 34 h. Computing the reduced space $\boldsymbol{V}$ of size $\sim$ 3650 $\times$ 1100 and the DEIM training data and approximation were the most time consuming operations with $\sim$ 95 h and $\sim$ 60h.

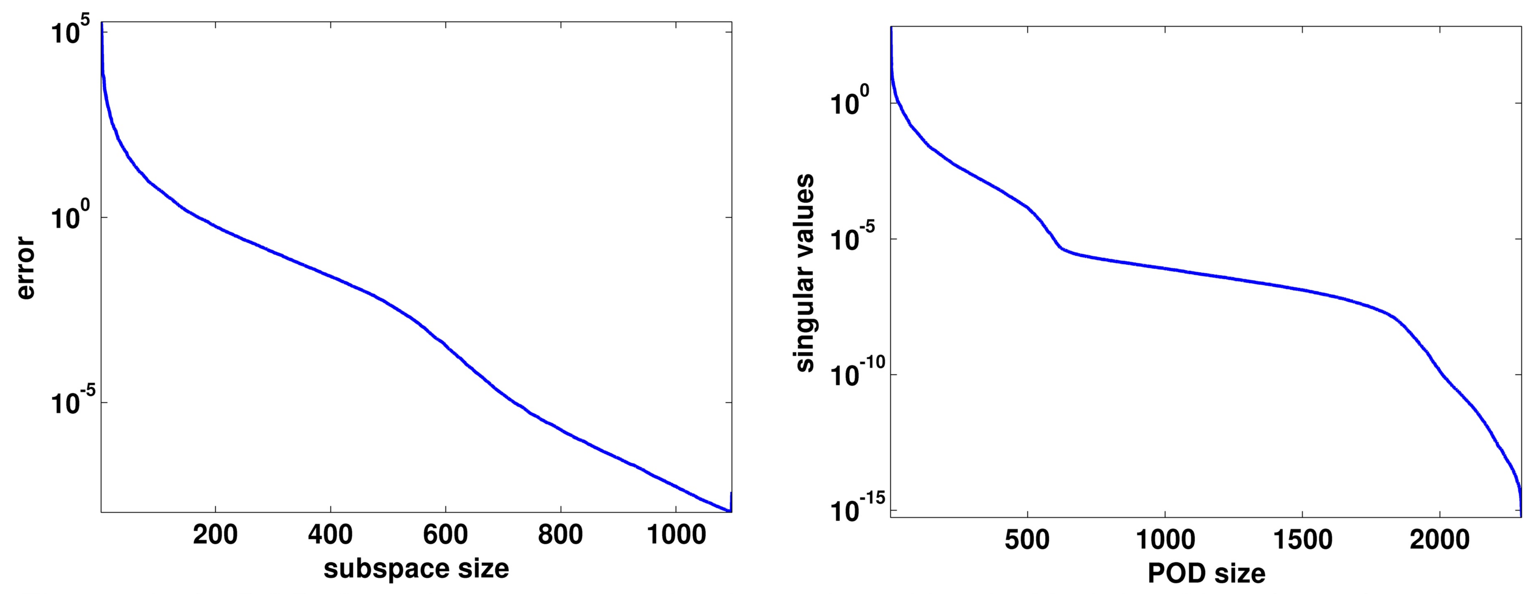

Figure: Left: POD-Greedy error decay for the reduced basis $\boldsymbol{V}$, right: Singular value decay for the DEIM basis $\boldsymbol{U}$.

The singular value decays for both, $\boldsymbol{V}$ and $\boldsymbol{U}$ look promising and solutions obtained by the reduced model are sufficiently accurate. However, the reduced model is not stable, i.e. a solution cannot be obtained for some parameter combinations due to convergence problems. This phenomenon is less pronounced when using a POD reduction only and evaluating the nonlinear part depending on the full dimension d. Furthermore, the required size of the reduced model and the required DEIM order are still too large, thus no computational speedup could be obtained so far.

Outlook

Different approaches shall be investigated to improve the stability and the computation time of the reduced model. This could include

- the computation of the POD basis $\boldsymbol{V}$ requiring optimality in a different norm, e.g. the norm induced by the mass matrix $\boldsymbol{M}$, or the generalised stiffness matrix $\boldsymbol{K}$,
- the application of DAE theory to transform the Hessenberg index 2 DAE into an ODE on a constraint manifold, and
- a choice of DEIM points optimised for systems originating from the finite element method (e.g. UDEIM).