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Testing for Wilson's quantum field theory in less than 4 dimensions

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ABSTRACT

Wilson's quantum field theory (QFT) in less than 4 dimensions has achieved a great success in the study of critical phenomenon but is still not tested within the scope of particle physics. To guarantee the validity of Wilson's QFT in less than 4 dimensions, Newton–Leibniz's differential-integral formulas must be extended to the noninteger dimensional situation. We show that this leads to a new prediction that Planck's constant will be expressed in terms of three fundamental constants: critical time scale, dimension of time axis, and total energy of universe. We propose the corresponding methods to measure these three constants. It will be thus interesting to compare the well-known value of Planck's constant with the potential theoretical value consisting of three fundamental constants.

The Higgs particle is an important part of the standard model of particle physics, and it has been recently discovered at CERN [1, 2]. The Higgs field is usually taken to be a scalar field with quartic self-interaction, i.e., $\lambda\phi^4$, so that spontaneous symmetry breaking can occur. Unfortunately, such a quartic term has been overshadowed by the issue of "triviality"; that is, λ vanishes in the limit of infinite cutoff [3]. To eliminate the triviality, Wilson [3] proposed that the quantum field theory (QFT) should be established in the D-dimensional space-time, where $D = 4 - \eta$ with η small. Wilson's proposal is a crazy idea. To avoid the awkward situation of fractional-dimensional space-time, Halpern and Huang [4, 5] searched for alternatives to the trivial $\lambda\phi^4$ field theory by considering nonpolynomial potentials. They found that the existence of nontrivial fixed points (i.e., ultraviolet fixed points) is indeed associated with the nonpolynomial potentials. Halpern and Huang's theory won some supports [6] but was still questioned [7].

Wilson's proposal implies that we may live in a fractional-dimensional universe where $\lambda\phi^4$ field theory is nontrivial. Although Wilson's proposal seems crazy, there have been many works supporting the viewpoint of fractal universe [8–15]. Most importantly, the proposal has been confirmed to be successful in the study of critical phenomenon [16, 17]. However, Wilson's QFT in less than 4 dimensions is still not tested in particle physics. The main purpose of this paper is just to propose some methods for testing the validity of Wilson's proposal within the framework of particle physics. To this end, let us remind that Wilson's QFT in less than 4 dimensions lacks rigid mathematical foundation. For example, Wilson [3] conjectured an integral formula for dealing with fractional-dimensional manifold but left its rigid proof to mathematicians. As a result, Wilson could not provide the corresponding derivative formula, namely the inverse transformation of his fractional-dimensional integral formula. Remarkably, Svozil [18] and Tao [19] have improved Wilson's QFT in less than 4 dimensions by generalizing Newton–Leibniz's differential-integral formulas to the fractional-dimensional situation. This paper points out that such an extension will lead to new prediction that can be tested.

THE VALIDITY OF QFT IN LESS THAN 4 DIMENSIONS

To obtain the QFT in less than 4 dimensions, Wilson [3] conjectured that the integral of any spherically symmetric function $f(r)$ on a D-dimensional space-time Ω should yield:

$$\int f(r)d\mu_H = \frac{2\pi^{D/2}}{\Gamma(D/2)}\int_0^R f(r)r^{D-1}dr, \tag{1}$$

where, $d\mu_H = d\mu_H(\Omega)$ denotes the local Hausdorff measure of Ω and D is a fraction (e.g., D = 3.99 in reference 3). Unfortunately, by Hausdorff measure $d\mu_H(\Omega) = \prod_{j=1}^n (\Delta x_j)^{D_j}$, one cannot arrive at the formula (1), where $D = \Sigma_{j=1}^n D_j$. Even so, Svozil [18] pointed out that the formula (1) would be valid provided that the local Hausdorff measure $d\mu_H(\Omega)$ was compelled to equal the product of n differentials (or differences) with different orders $D_1, D_2, \ldots, D_n$; that is,

$$d\mu_H(\Omega) = \prod_{j=1}^n (\Delta x_j)^{D_j} = \prod_{j=1}^n d^{D_j}x_j, \tag{2}$$

where, d^{D_j} denotes the differential operator of order D_j.

Motivated by Svozil's formula (2), Tao [19] further suggested that one could replace Hausdorff measure $(\Delta x_j)^{D_j}$ by fractional differential $d^{D_j}x_j$ to describe the local length of any D_j-dimensional curve. According to Tao's suggestion, if $x(l-j\Delta l)$ represents the point on a m-dimensional fractal curve and $j = 0, 1, 2, ...$, then the distance between points $x(l)$ and $x(l-\Delta l)$ should yield:

$$|\Delta_m[x(l), x(l-\Delta l)]| = \left|\sum_{j=0}^{\infty} \frac{m(m-1)\cdots(m-j+1)(-1)^j}{j!} x(l-j\Delta l)\right|, \tag{3}$$

where, Δ_m denotes the difference operator of order m and one has:

$$\lim_{\Delta l\to 0} \Delta_m[x(l), x(l-\Delta l)] = d^m x(l)$$

Obviously, Tao's fractional-dimensional measure (3) will return to the Euclidean measure $|x(l) - x(l-\Delta l)|$ whenever the dimension of the fractal curve, m, equals 1, that is,

$$|\Delta_{m=1}[x(l), x(l-\Delta l)]| = |x(l) - x(l-\Delta l)|.$$

Because the Euclidean measure $|x(l) - x(l-\Delta l)|$ has been extended to the fractional-dimensional measure $|\Delta_m[x(l), x(l-\Delta l)]|$, Tao [19] introduced the corresponding "fractal differential-integral formulas" as well.

If one denotes by $x = x(l)$ the length of a m-dimensional fractal curve $\beta_m(l)$ (by the fractal theory $x(l)\sim l^m$), and if the function $f(x)$ is defined on the fractal curve $\beta_m(l)$, then the *fractal derivative* of $f(x)$ can be written in the form [19]:

$$\frac{D_l^m f(x)}{D_l^m x} = \lim_{\Delta l\to 0} \frac{\Delta_m[\bar{f}(l), \bar{f}(l-\Delta l)]}{\Delta_m[x(l), x(l-\Delta l)]} = \frac{d^m \bar{f}(l)/dl^m}{d^m x(l)/dl^m}, \tag{4}$$

where, $\bar{f}(l) = f[x(l)]$.

Obviously, the fractal derivative (4) will return to the well-known Newton–Leibniz's derivative whenever $m = 1$. It is carefully noted that Tao's fractal derivative (4) is different from the conventional fractional derivative, e.g. Riemann–Liouville fractional derivative and Caputo fractional derivative. To see this, we only need to acknowledge that by Tao's definition [19] $d^m\bar{f}(l)/dl^m$ or $d^m x(l)/dl^m$ in the fractal derivative (4) is just the Riemann–Liouville fractional derivative. In fact, neither Riemann–Liouville fractional derivative nor Caputo fractional derivative is the inverse transformation of Wilson's fractional-dimensional integral formula (1). In other words, Riemann–Liouville fractional integral and Caputo fractional integral are both different from Wilson's integral formula (1). This is just why Tao introduced the fractal derivative (4) in reference 19. Remarkably, we will immediately see that Tao's fractal derivative (4) can be regarded as an inverse transformation of Wilson's fractional-dimensional integral formula (1). To this end, we need to introduce Tao's fractal integral.

By the fractal derivative (4), Tao's *fractal integral* is defined in the form [19]:

$$\int D_l^m f(x) = \int \frac{D_l^m f(x)}{D_l^m x} D_l^m x \tag{5}$$

Undoubtedly, $D_l{}^m x$ can be thought of as the differential element of the m-dimensional fractal curve $\beta_m(l)$.

Specifically, by the formula (5), the validity of the formula (1) can be verified. To see this, let us consider four fractal curves $\beta_{D_i}(l_i)$ with the dimensions D_i respectively, where $i = 0, 1, 2, 3$. If we denote by $\beta_{D_0}(l_0)$ the time axis and denote by $\beta_{D_j}(l_j)$ the space axes ($j = 1, 2, 3$), then the Cartesian product of $\beta_{D_i}(l_i)$ can produce a D-dimensional space-time Ω, where $D = D_0 + D_1 + D_2 + D_3$ and $\Omega = \beta_{D_0}(l_0) \otimes \beta_{D_1}(l_1) \otimes \beta_{D_2}(l_2) \otimes \beta_{D_3}(l_3)$. If the spherically symmetric function $f(l)$ is defined on Ω, then by (5) Tao [19] proved the following formula:

$$\int\int\int\int f(l) \prod_{i=0}^{3} D_{l_i}^{D_i} x_i = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^R f(l) l^{D-1} dl, \tag{6}$$

where, $l^2 = l_0^2 + l_1^2 + l_2^2 + l_3^2$.

The formula (6) approves that the formula (1) is valid for any real number D. In other words, Wilson's QFT in less than 4 dimensions will be available when the fractal differential-integral formulas (4) and (5) are admitted.

NEW PREDICTION

The fractal differential-integral formulas (4) and (5) can be thought of as the extension of Newton–Leibniz's differential-integral formulas in the noninteger dimensional manifold, and such an extension will lead to new prediction that can be tested. To see this, we need to introduce two assumptions as follows:

Assumption (i). The total energy of the universe $E(t)$ is independent of time variable t, that is, $E(t) = E = const$.

Assumption (ii). The dimension of time axis $D_0 = \omega$ is slightly less than 1.

The Assumption (i) is natural and necessary; it indicates that the total energy of universe is conserved. It is here worth mentioning that the Assumption (i) has been supported by the standard model of modern cosmology. For details, see the discussions in Measurement 1, where we will see that a constant can be defined by specifying zero integer derivative.

The Assumption (ii) will guarantee that $D = \omega + D_1 + D_2 + D_3 < 4$. In particular, if $D_1 = D_2 = D_3 = 1$, then $4 - D$ will be a small quantity, so that the Feynman graph can be expanded in powers of $4 - D$. This is just the starting point of Wilson's QFT in less than 4 dimensions [3].

Now we show that by the fractal differential-integral formulas (4) and (5) the Assumptions (i) and (ii) will lead to Planck's energy quantum. By the Assumption (ii), the time axis is a ω-dimensional fractal curve; therefore, by the formula (4) and the Assumption (i), the fractal derivative of $E(t)$ with respect to the time axis $t \sim l^{\omega}$ follows (see the formula (B.1) in reference 19):

$$D_l^{\omega}E(t)/D_l^{\omega}t = D_l^{\omega}E/D_l^{\omega}t = \frac{E}{\Gamma\,(1-\omega)\,\Gamma\,(1+\omega)} \cdot \frac{1}{t}. \tag{7}$$

It is easy to compute:

$$\lim_{\omega \to 1} \frac{1}{\Gamma(1-\omega)\,\Gamma\,(1+\omega)} \approx 1-\omega. \tag{8}$$

By the Assumption (ii), we have $1 - \omega \to 0$, so substituting (8) into (7) yields:

$$D_l^{\omega}E/D_l^{\omega}t \approx E \cdot (1-\omega) \cdot (1/t). \tag{9}$$

In particular, when $\omega = 1$, the equation (9) gives the conventional Newton–Leibniz's derivative:

$$D_l^{\omega=1}E/D_l^{\omega=1}t = dE/dt = 0. \tag{10}$$

To see the physical meaning of the equation (9), one can rewrite it in the form:

$$D_l^{\omega}E \approx E \cdot (1-\omega) \cdot (1/t) \cdot D_l^{\omega}t. \tag{11}$$

On the other hand, by the formulas (5) and (7), one has:

$$E = \int D_l^{\omega}E, \tag{12}$$

where we have used the formula of fractional integral [20]

$$\begin{aligned}\int l^{-\omega}(dl)^{\omega} &= \frac{1}{\Gamma(\omega)}\int_0^t (l-z)^{\omega-1} z^{-\omega}dz \\ &= \Gamma(1-\omega), \qquad (0<\omega<1)\end{aligned}$$

and the formula of fractional derivative [20]

$$d^m l^n/dl^m = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} l^{n-m}.$$

The equation (12) indicates that the total energy of the universe, E, equals the sum of all the energy quanta $D_l^{\omega}E$.
Let us order

$$h = E \cdot (1-\omega) \cdot \Delta T, \tag{13}$$

$$\nu = 1/t, \tag{14}$$

$$\Delta T = D_l^{\omega}t, \tag{15}$$

$$\varepsilon = D_l^{\omega}E, \tag{16}$$

where ΔT denotes the critical time scale which can be regarded as a natural cutoff of the renormalization theory. Following Wilson's spirit, "any quantum field theory is defined fundamentally with a cutoff Λ that has some physical significance. In statistical mechanical applications, this momentum scale is the inverse atomic spacing. In QFT appropriate to elementary particle physics, the cutoff would have to be associated with some fundamental graininess of space-time", see page 402 in reference 21. In our opinion, $\Delta T \sim 1/\Lambda$ can be just thought of as the fundamental graininess of time.
Substituting (13)–(16) into (11), we obtain Planck's energy quantum formula:

$$\varepsilon = h\nu, \tag{17}$$

where h denotes Planck's constant and ν the frequency.
By de Broglie's method, the formula (17) will inevitably lead to Matter wave $\psi(t, x)$. If the dimension of time axis, ω, approaches 1 very near, Tao's fractal derivative (4) can be approximately replaced by Newton–Leibniz's derivative. As a result, by Schrodinger's procedure, the traditional quantum mechanics can be derived. Later, by available data, we shall estimate the approximate value of ω which approaches $1-10^{-59}$. This is why our traditional quantum mechanics works so fine in precisely 4-dimensional space-time. However, traditional quantum mechanics is only an asymptotic form of fractional-dimensional quantum mechanics.
The formula (13) is the core result of this paper. It can be regarded as a new prediction of Wilson's QFT in less than 4 dimensions. This means, we may judge the validity of QFT in less than 4 dimensions by testing the formula (13). By the formula (13), Planck's constant is completely determined by critical time scale ΔT, dimension of time axis ω, and total energy of universe E. Therefore, the remainder of this paper will introduce the methods of measuring ΔT, ω, and E.

TESTING METHODS

Measurement 1 – we show how to measure the total energy of the universe, E. For convenience, the speed of light c is set equal to 1.
To measure the value of E, we must first make clear what is the total energy of the universe. As is well known, by the Robertson–Walker metric, the conservation of energy of Einstein's gravitation equation leads to [22]: $\rho_\sigma R^{3(1+\sigma)} = const$, where ρ_σ denotes the total energy density of the universe and R denotes the scale of the universe. Therefore, we can simply denote by $\rho_\sigma R^{3(1+\sigma)}$ the total energy of the universe which agrees with the Assumption (i).
Specifically, $\sigma = 1/3$ describes the early universe which is known as radiation-dominated, and meanwhile $\sigma = 0$ describes the universe which is known as matter-dominated. We believe that today the energy density of the universe is dominated by matter [23], so we will adopt $\sigma = 0$. Thus, the total energy of the universe, $\rho_\sigma R^{3(1+\sigma)}$, can be approximately interpreted as:

$$E \approx \rho_0 R^3. \tag{18}$$

Now we show how to determine ρ_0 and R. According to the standard model of modern cosmology, one has [22]:

$$\rho_0/\rho_c - 1 = k/H^2R^2, \tag{19}$$

and

$$\rho_c = 3H^2/8\pi G, \tag{20}$$

where k denotes the curvature of the universe, ρ_c denotes the critical density of the universe, H denotes Hubble's constant, and G denotes Newton's constant.
Recently, via the measurement of the cosmic microwave background radiation, de Bernardis et al. [24] discovered that our universe should be flat, that is, $k \approx 0$. Then by (19) one has:

$$\rho_0 \approx \rho_c. \tag{21}$$

Because the value of ρ_c is known, if one can find the current value of R, then by (18) one will determine the value of E. Fortunately, R follows the definition of Hubble's constant:

$$dR/dt = H \cdot R. \tag{22}$$

Thus, by (22) one can determine the current value of R via measuring the age of the universe.
Remarkably, the Wilkinson Microwave Anisotropy Probe (WMAP) has brought the crucial data [25] for estimating the total mass of our universe. The corresponding data have been listed in Table 1. Using these available data, Geherels [26] has estimated the total mass of our universe: see Table 2.
Measurement 2 – we show how to measure ω and ΔT.
Considering that the time axis is a fractal curve, we shall propose the following three steps to measure the length of a time interval, L.

(1) One uses a standard clock to specify a time interval whose length is L. For example, the length L can be denoted by a step length of the standard clock.
(2) One finds another clock i whose step length is L/N_i (less than the step length of the standard clock), where $N_i > 1$.
(3) Let the standard clock and the clock i go simultaneously. Once the standard clock arrives at L from 0, one immediately counts the number of steps, $M_i = M_i(N_i)$, that the clock i has gone.

Table 1. Observed values of age and density of universe [25]

Age of universe (year)	1.373(+0.013, −0.017) $\times 10^{10}$
Baryon-density of universe ($kg \cdot m^{-3}$)	4.19(+0.13, −0.17) $\times 10^{-28}$
Critical density of universe ($kg \cdot m^{-3}$)	4.0(±0.4) $\times 10^{-28}$

Table 2. Total mass of universe [26].

	Total mass of universe (Proton masses)
Baryon-density of universe	2.300(+0.096, −0.126) $\times 10^{78}$
Critical density of universe	2.2(±0.2) $\times 10^{78}$
Chandrasekhar method	1.13179 $\times 10^{78}$

Since the dimension of the time axis equals ω, then by Mandelbrot's fractal theory [27], one has:

$$\lim_{N_i \to \infty} (L/N_i)^{\omega} \cdot M_i = L^{\omega}. \tag{23}$$

The equation (23) shows that when one uses the clock i to measure "L", then the value of "L" equals L^{ω}, where $\omega = lim_{N_i \to \infty} \frac{lnM_i}{lnN_i}$.
In an actual measurement, one can use the clocks with different step lengths to approach the limit $N_i \to \infty$, that is,

$$N_1 < N_2 < \cdots < N_i < \cdots < N_n. \tag{24}$$

Of course, since there exists a critical time scale ΔT, the actual step length of the clock i cannot approach zero; that is to say, the smallest step length is restricted by:

$$\lim_{N_i \to \infty} (L/N_i + \Delta T) = \Delta T. \tag{25}$$

As a result, the equation (23) must be revised in the form:

$$(L/N_i + \Delta T)^{\omega} \cdot M_i = L^{\omega}, \tag{26}$$

for $i = 1, 2, ..., n$.
Taking the logarithm on both sides of (26), one has:

$$\omega \cdot ln\left(\frac{1}{N_i} + \delta\right) = ln\left(\frac{1}{M_i}\right), \tag{27}$$

where, $\delta = \Delta T/L$.
Since ΔT or δ lies far beyond the reach of present-day experiments, one can have $\delta \ll 1/N_i$ for any i; therefore, one has the following Taylor expansion:

$$ln\left(\frac{1}{N_i} + \delta\right) \approx ln\left(\frac{1}{N_i}\right) + N_i \cdot \delta. \tag{28}$$

Substituting (28) into (27) yields:

$$\frac{lnM_i}{N_i} = -\omega \cdot \delta + \omega \cdot \left(\frac{lnN_i}{N_i}\right). \tag{29}$$

If we order $y_i = \frac{lnM_i}{N_i}, x_i = \frac{lnN_i}{N_i}$, and $\beta = -\omega \cdot \delta$, then the equation (29) can be written in the form:

$$y_i = \beta + \omega \cdot x_i. \tag{30}$$

Obviously, using the measured value (M_i,N_i) one can compute (x_i,y_i), where $i = 1, 2, ..., n$. Then by the least square method we have the following estimated values:

$$\omega = \left[\sum\nolimits_{i=1}^{n} x_i y_i - (1/n) \cdot \left(\sum\nolimits_{i=1}^{n} x_i\right) \cdot \left(\sum\nolimits_{i=1}^{n} y_i\right)\right] \Big/ \left[\sum\nolimits_{i=1}^{n} x_i^2 - (1/n) \cdot \left(\sum\nolimits_{i=1}^{n} x_i\right)^2\right] \tag{31}$$

(32) $$\beta = (1/n) \cdot (\sum_{i=1}^{n} y_i) - (\omega/n) \cdot (\sum_{i=1}^{n} x_i).$$

As long as ω and β are found, by the formulas $\delta = \Delta T/L$ and $\beta = -\omega \cdot \delta$, one can obtain the estimated value of ΔT. Finally, using the estimated values of ΔT, ω, and E, one can compute the theoretical value of Planck's constant.

CONCLUSION

In summary, our formula (13), $h = E \cdot (1 - \omega) \cdot \Delta T$, can be tested in principle. On the one hand, the Table 2 has listed the observed value of total mass of our universe which is approximately denoted by 10^{51}kg. On the other hand, the critical time scale ΔT can be thought of as the fundamental graininess of time and therefore plays the similar role with the Planck time. If we roughly denote the critical time scale ΔT by the Planck time which yields 5×10^{-44} s, then by the formula (13) we can estimate the approximate value of $1 - \omega$ which should approach 10^{-59}. Since the dimension of time axis, ω, approaches 1 so near, it will be difficult to distinguish Wilson's QFT in less than 4 dimensions from current QFT. However, we believe that the difference between ω and 1 may emerge in the small scale. Therefore, we suggest doing the Measurement 2 using the atomic clocks with different step lengths. We strongly suggest completing the Measurement 2. If the measured value of ω is in agreement with our estimated value, then Wilson's QFT in less than 4 dimensions will pass the test. Not only so, we will be able to explain the origin of quantum behaviors as well. This is because, according to our analysis, the energy quantum formula (13) is due to $\omega < 1$ which implies a fractional-dimensional universe. In other words, quantum behaviors arise because our universe is a fractal.

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COMPETING INTERESTS

The author declares no competing financial interests.

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