SVD on Intersection Spaces

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Introduction

I. For a linear, bounded and compact operator $T \in \mathcal{K}(H_2, H_1)$ between Hilbert spaces H_2 and H_1 the spectral theorem tells us

$$(SVD) \quad Tw = \sum_{k=1}^{\infty} \sigma_k \langle w, h_k^2 \rangle_{H_2} h_k^1, \quad \forall w \in H_2 \quad (1)$$

for orthonormal $\{h_k^1\}_k, \{h_k^2\}_k$ and positive non-increasing $\{\sigma_k\}$ with $\lim_k \sigma_k = 0$.

The space $H = (H_1 \otimes H_2, \langle \cdot, \cdot \rangle_H)$ equipped with the canonical inner product

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_H = \langle v_1, v_2 \rangle_{H_1} \cdot \langle w_1, w_2 \rangle_{H_2} \quad (2)$$

is a Hilbert space. An algebraic tensor $u = \sum_{k=1}^r v_k \otimes w_k \in H_1 \otimes_a H_2$ can be identified with an operator via

$$T_u(w) = \sum_{k=1}^r \langle w, w_k \rangle_{H_2} v_k.$$

Property (2) implies $\left(\{h_k^1\}_k \text{ and } \{h_k^2\}_k \text{ orthonormal} \Rightarrow \{h_k^1 \otimes h_m^2\}_{k,m} \text{ orthonormal} \right)$. Then, $\|u\|_H = \|T_u\|_{HS}$ and $H = HS(H_2, H_1)$.

SVD from (1) applied to u provides an **optimal low rank approximation**. Example: $H = L_2(\Omega_1 \times \Omega_2)$.

II. For $H = H^1(\Omega_1 \times \Omega_2)$

$$H^1(\Omega_1 \times \Omega_2) \not\subset \mathcal{K}(H^1(\Omega_2), H^1(\Omega_1)).$$

Here **SVD does not apply**.

The Sobolev H^1 is isomorphic to the intersection space

$$\begin{aligned} H^1(\Omega_1 \times \Omega_2) &= (H^1 \otimes L_2) \cap (L_2 \otimes H^1) \\ &=: H^{10} \cap H^{01}. \end{aligned}$$

SVD applies on both H^{10} and H^{01} . For $u \in H^1$

$$\begin{aligned} u &= \sum_{k=1}^{\infty} \sigma_k^{10} x_k^1 \otimes y_k^0 \quad \text{in } \|\cdot\|_{10}, \\ u &= \sum_{k=1}^{\infty} \sigma_k^{01} x_k^0 \otimes y_k^1 \quad \text{in } \|\cdot\|_{01}. \end{aligned}$$

For details see [1].

Theoretical Perspective

I. Consider the finite dimensional subspaces

$$U_r^{10} := \text{span}(x_k^1 : 1 \leq k \leq r), \quad U_r^{01} := \text{span}(y_k^1 : 1 \leq k \leq r),$$

and the H^1 orthogonal projections $P_r : H^1(\Omega_1) \rightarrow U_r^{10}$ and $Q_r : H^1(\Omega_2) \rightarrow U_r^{01}$. One can show

Theorem 1. Assume additionally $u \in H^2(\Omega_1 \times \Omega_2)$. This implies $U_r^{10} \subset H^2(\Omega_1)$ and $U_r^{01} \subset H^2(\Omega_2)$. Define the regularity measures

$$L(r) = \sup_{v \in U_r^{10}} \frac{\|v\|_1}{\|v\|_0} \cdot \sup_{v \in U_r^{10}} \frac{\|v\|_2}{\|v\|_1}, \quad R(r) = \sup_{v \in U_r^{01}} \frac{\|v\|_1}{\|v\|_0} \cdot \sup_{v \in U_r^{01}} \frac{\|v\|_2}{\|v\|_1}. \quad (3)$$

Then, we get

$$\|u - (P_r \otimes Q_r)u\|_1 \leq \left(\sum_{k=r+1}^{\infty} (1 + rL^2(r))(\sigma_k^{10})^2 + (1 + rR^2(r))(\sigma_k^{01})^2 \right)^{1/2}.$$

More generally, for $d > 2$, applying HOSVD ([2]) w.r.t. (a generalized) $\|\cdot\|_{mix}$ -norm

$$\|u - (P_r^1 \otimes \cdots \otimes P_r^d)u\|_1 \leq (\sqrt{r}C(r))^{d-2} \left(\sum_{j=1}^d \sum_{k=r+1}^{\infty} (\sigma_k^j)^2 \right)^{1/2}.$$

II. Applying SVD to coefficients. Consider a 2π -periodic function $u \in H^1([-\pi, \pi]^2)$. Then

$$\begin{aligned} u &= \frac{1}{2\pi} \sum_{k,m \in \mathbb{Z}} c_{km} e^{ikx} e^{imy} = \frac{1}{2\pi} \sum_{k,m \in \mathbb{Z}} c_{km} \sqrt{1+k^2+m^2} \frac{1}{\sqrt{1+k^2+m^2}} e^{ikx} e^{imy} \\ &\approx \frac{1}{2\pi} \sum_{l=1}^{\infty} \sigma_l \sum_{\nu \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} E(k, \nu) v_k^l e^{ikx} \right) \left(\sum_{m \in \mathbb{Z}} E(m, \nu) w_m^l e^{imy} \right), \end{aligned}$$

where E are exponentials. Similar was applied in [3].

III. Consider the direct sum space $H_{2D} := H^{10} \times H^{01}$. We get a natural interpretation for $u \in H^1$.

Theorem 2. The Sobolev space $H^1 \hookrightarrow H_{2D}$ can be isometrically embedded into H_{2D} via $u \mapsto (u, u)$. This gives the best low rank approximation as

$$\inf_{\substack{v \in H^{10}, \text{rank}(v) \leq r, \\ w \in H^{01}, \text{rank}(w) \leq r}} \|u - (v, w)\|_{2D} = \|u - ((P_r \otimes id)u, (id \otimes Q_r)u)\|_{2D}.$$

Moreover, interpreting u as an operator in $HS(L_2(\Omega_1) \times H_1(\Omega_2), H^1(\Omega_1) \times L_2(\Omega_2))$ we obtain the SVD

$$u = \sum_{k=1}^{\infty} \sigma_k^{\cup} x_k \otimes y_k, \quad x_k \otimes y_k = \begin{cases} (x_l^1, 0) \otimes (y_l^0, 0), & \text{if } \sigma_k^{\cup} = \sigma_l^{10}, \\ (0, x_l^0) \otimes (0, y_l^1), & \text{if } \sigma_k^{\cup} = \sigma_l^{01}. \end{cases}$$

References

- [1] W. Hackbusch. Tensor spaces and numerical tensor calculus. *Springer Series in Computational Mathematics*, 42:xxiv+500, 2012.
- [2] L. Grasedyck. Hierarchical singular value decomposition of tensors. *SIAM J. Matrix Anal. Appl.*, 31:2029-2054, 2009/10.
- [3] M. Bachmayr, W. Dahmen. Adaptive low-rank methods: Problems on Sobolev spaces. *SIAM J. Numer. Anal.*, 54:744-796, 2016.

Acknowledgements

The collaboration between Mazen Ali and Anthony Nouy was supported by the European Model Reduction Network (EU-MORNET).

Numerical Experiments

Figure 1: Low rank approximations for truncated Fourier series of $u(x, y) = (x^2 + y^2)^{0.3}$ (1,1) and $u(x, y) = |x + y|^{0.6}$ (1,2). Galerkin solutions for reference functions $u(x, y) = \exp(\cos(x) \cos(y))$ (2,1) and $u(x, y) = |1 - (x^2 + y^2)|^{0.95}$ (2,2).