Schur Products
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ABSTRACT
We call Schur products (in analogy to crossed products which inspired the present construction) a C*-algebra obtained from a given unital C*-algebra $E$ and a discrete group $T$. This new C*-algebra consists of operators acting on the Hilbert right $E$-module $E \otimes \ell^2(T)$ and are defined by a twisted representation of $T$.

INTRODUCTION, NOTATION, AND TERMINOLOGY
The projective representation of groups was introduced in 1904 by Issai Schur (1875–1941) in his paper [S]. It differs from the normal representation of groups (introduced by his tutor Ferdinand Georg Frobenius (1849–1917) at the suggestion of Richard Dedekind (1831–1916)) by a twisting factor, which we call Schur function in this article and which is called sometimes normalized factor set in the literature (other names are also used). It starts with a discrete group $T$ and a Schur function $f$ for $T$. This is a scalar valued function on $T \times T$ satisfying the conditions $f(1,1) = 1$ and $j f(s,t) j^{-1} = f(r,s)f(r,t)f(s,t)$ for all $r, s, t \in T$. The projective representation of $T$ twisted by $f$ is a unital C*-subalgebra of the C*-algebra $L(\ell^2(T))$ of operators on the Hilbert space $\ell^2(T)$. This representation can be used in order to construct many examples of C*-algebras (see e.g. [1] Chapter 7). By replacing the scalars $\mathbb{R}$ or $\mathbb{C}$ with an arbitrary unital (real or complex) C*-algebra $E$ the field of applications is enhanced in an essential way. In this case $\ell^2(T)$ is replaced by the Hilbert right $E$-module $E \otimes \ell^2(T)$ and $L(\ell^2(T))$ is replaced by $L(E \otimes \ell^2(T))$, the C*-algebra of adjointable operators of $L(E \otimes \ell^2(T))$. We call Schur product of $E$ and $T$ the resulting C*-algebra. It opens a way to create new K-theories ([4]).

In a preliminaries we introduce some results which are needed for this construction, which is developed in the schur products. In the examples we present examples of C*-algebras obtained by this method. The classical Clifford Algebras (including the infinite dimensional ones) are C*-algebras which can be obtained by projective representations of certain groups ([1] Section 7.2). The clifford algebras of this article is dedicated to the generalization of these Clifford Algebras as an example of Schur products.

Throughout this article we use the following notation: $T$ is a group, 1 is its neutral element, $K := \ell^2(T)$, $1_K := id_K :=$ identity map of $K$. $E$ is a unital C*-algebra (resp. a W*-algebra), $1_E$ is its unit, $\hat{E}$ denotes the set $E$ endowed with its canonical structure of a Hilbert right $E$-module ([1] Proposition 5.6.1.5),

$$H := \hat{E} \otimes K \cong \bigoplus_{t \in T} \hat{E}, \quad (\text{resp. } H := \hat{E} \otimes K \cong \bigoplus_{t \in T} \hat{E})$$

([3] Proposition 2.1, (resp. [3] Corollary 2.2)). In some examples, in which $T$ is additive, 1 will be replaced by 0.
The map 
\[ \mathcal{L}_E(\hat{E}) \rightarrow E, \; u \mapsto \langle u1_E | 1_E \rangle = u1_E \]
is an isomorphism of C*-algebras with inverse 
\[ E \rightarrow \mathcal{L}_E(\hat{E}), \; x \mapsto x. \]

We identify \( E \) with \( \mathcal{L}_E(\hat{E}) \) using these isomorphisms.

In general we use the notation of [1]. For tensor products of C*-algebras we use [9], for W*-tensor products of W*-algebras we use [8], for tensor products of Hilbert right C*-modules we use [6], and for the exterior W*-tensor products of selfdual Hilbert right W*-modules we use [2] and [3].

In the sequel we give a list of notation (mainly introduced in [1]) which are used in this article.

1) \( \mathbb{K} \) denotes the field of real numbers (= \( \mathbb{R} \)) or the field of complex numbers (= \( \mathbb{C} \)). In general the C*-algebras will be complex or real. \( \mathbb{H} \) denotes the field of quaternions, \( \mathbb{N} \) denotes the set of natural numbers (\( 0 \notin \mathbb{N} \)), and for every \( n \in \mathbb{N} \cup \{ 0 \} \) we put
\[ \mathbb{N}_n := \{ m \in \mathbb{N} \mid m \leq n \}. \]
\( \mathbb{Z} \) denotes the group of integers and for every \( n \in \mathbb{N} \) we put \( \mathbb{Z}_n := \mathbb{Z} / (n\mathbb{Z}) \).

2) For every set \( A \), \( \Psi(A) \) denotes the set of subsets of \( A \), \( \Psi_f(A) \) the set of finite subsets of \( A \), and \( \text{Card} \; A \) denotes the cardinal number of \( A \). If \( f \) is a function defined on \( A \) and \( B \) is a subset of \( A \) then \( f[B] \) denotes the restriction of \( f \) to \( B \).

3) If \( A, B \) are sets then \( A^B \) denotes the set of maps of \( A \) in \( B \).

4) For all \( i, j \) we denote by \( \delta_{ij} \) Kronecker’s symbol:
\[ \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \]

5) If \( A, B \) are topological spaces then \( C(A, B) \) denotes the set of continuous maps of \( A \) into \( B \). If \( A \) is locally compact space and \( E \) is a C*-algebra then \( C(A, E) \) (resp. \( C_0(A, E) \)) denotes the C*-algebra of continuous maps \( A \rightarrow E \), which are bounded (resp. which converge to 0 at the infinity).

6) For every set \( I \) and for every \( J \subset I \) we denote by \( e_j := e_j^I \) the characteristic function of \( J \) i.e. the function on \( I \) equal to 1 on \( J \) and equal to 0 on \( I \setminus J \). For \( i \in I \) we put \( e_i := (\delta_{ij})_j \in \ell^2(I) \).

7) If \( F \) is an additive group and \( S \) is a set then
\[ F^S := \{ x \in F^S \mid \text{satisfies } x_s \neq 0 \} \text{ is finite}. \]

8) If \( E, F \) are vector spaces in duality then \( E_F \) denotes the vector space \( E \) endowed with the locally convex topology of pointwise convergence on \( F \), i.e. with the weak topology \( \sigma(E, F) \).

9) If \( E \) is a normed vector space then \( E^* \) denotes its dual and \( E^u \) denotes its unit ball:
\[ E^u := \{ x \in E \mid \|x\| \leq 1 \}. \]

Moreover if \( E \) is an ordered Banach space then \( E^+ \) denotes the convex cone of its positive elements. If \( E \) has a unique predual (up to isomorphisms), then we denote by \( \hat{E} \) this predual and so by \( E_\hat{E} \) the vector space \( E \) endowed with the locally convex topology of pointwise convergence on \( \hat{E} \).

10) The expressions of the form “\( \cdots C^\ast \cdots \) (resp. \( \cdots W^* \cdots \))”, which appear often in this article, will be replaced by expressions of the form “\( \cdots C^{\ast\ast} \cdots \)”.

11) If \( F \) is a unital C*-algebra and \( A \) is a subset of \( F \) then we denote by \( 1_F \) the unit of \( F \), by \( Pr \; F \) the set of orthogonal projections of \( F \), by
\[ A^\circ := \{ x \in F \mid y \in A \Rightarrow xy = yx \}, \quad \text{Re} \; F := \{ x \in F \mid x = x^\ast \}, \]
and by \( Un \; F \) the set of unitary elements of \( F \). If \( F \) is a real C*-algebra then \( \hat{F} \) denotes its complexification.

12) If \( F \) is a C*-algebra then we denote for every \( n \in \mathbb{N} \) by \( F_{n,n} \) the C*-algebra of \( n \times n \) matrices with entries in \( F \). If \( T \) is finite then \( F_{T,T} \) has a corresponding signification.
13) Let $F$ be a $C^*$-algebra and $H$, $K$ Hilbert right $F$-modules. We denote by $\mathcal{L}_F(H, K)$ the Banach subspace of $\mathcal{L}(H, K)$ of adjointable operators, by $1_H$ the identity map $H \rightarrow H$ which belongs to 

$$\mathcal{L}_F(H) = \mathcal{L}_F(H, H).$$

For $(\xi, \eta) \in H \times K$ we put

$$\eta(\cdot, \xi) : H \rightarrow K, \quad \zeta \mapsto \eta(\zeta, \xi)$$

and denote by $K_F(H)$ the closed vector subspace of $\mathcal{L}_F(H)$ generated by $\{\eta(\cdot, \xi) | \xi, \eta \in H\}$.

14) Let $F$ be a $W^*$-algebra and $H$, $K$ Hilbert right $F$-modules. We put for $a \in F$ and $(\xi, \eta) \in H \times K$,

$$(a, \xi) : H \rightarrow \mathbb{K}, \quad \zeta \mapsto \langle \zeta, \xi \rangle, a,$$

$$(a, \xi, \eta) : \mathcal{L}_F(H, K) \rightarrow \mathbb{K}, \quad u \mapsto \langle u \xi, \eta \rangle$$

and denote by $\bar{H}$ the closed vector subspace of the dual $H'$ of $H$ generated by

$$\{\langle x, \xi \rangle | a \in F, \xi \in H\}$$

and by $\hat{H}$ the closed vector subspace of $\mathcal{L}_F(H, K)'$ generated by

$$\{(a, \xi, \eta) | (a, \xi, \eta) \in F \times H \times K\}.$$

If $H$ is selfdual then $\bar{H}$ is the predual of $\mathcal{L}_F(H)$ ([1] Theorem 5.6.3.5 b)) and $\hat{H}$ is the predual of $H$ ([1] Proposition 5.6.3.3). Moreover a map defined on $F$ is called $W^*$-continuous if it is continuous on $F\bar{F}$. If $G$ is a $W^*$-algebra a $C^*$-homomorphism $\varphi : F \rightarrow G$ is called a $W^*$-homomorphism if the map $\varphi : F\bar{F} \rightarrow G\bar{G}$ is continuous; in this case $\varphi^*$ denotes the pretranspose of $\varphi$.

15) If $F$ is a $C^\star$-algebra and $(H_i)_{i \in I}$ a family of Hilbert right $F$-modules then we put

$$\bigoplus_{i \in I} H_i := \left\{ \xi \in \prod_{i \in I} H_i | \text{the family } \langle \xi_i | \xi_j \rangle_{i \in I} \text{ is summable in } F \right\}$$

respectively

$$\bigodot_{i \in I} H_i := \left\{ \xi \in \prod_{i \in I} H_i | \text{the family } \langle \xi_i | \xi_j \rangle_{i \in I} \text{ is summable in } F\bar{F} \right\}.$$

16) $\bigodot$ denotes the algebraic tensor product of vector spaces.

17) If $F$, $G$ are $W^*$-algebras and $H$ (resp. $K$) is a selfdual Hilbert right $F$-module (resp. G-module) then we denote by $H \bigodot K$ the $W^*$-tensor product of $H$ and $K$, which is a selfdual Hilbert right $H \bigodot G$-module ([2] Definition 2.3).

18) $\approx$ denotes isomorphic.

If $T$ is finite then (by [1] Theorem 5.6.6.1 f))

$$\mathcal{L}_E(H) = E^T_F = \mathbb{K}^T_T \bigodot E = \mathcal{K}_E(H).$$

PRELIMINARIES

1.1 Schur functions

We list in this subsection some properties of the Schur functions needed later.

**DEFINITION 1.1.1** A Schur $E$-function for $T$ is a map

$$f : T \times T \rightarrow \text{Un } E^c$$

such that $f(1, 1) = 1_E$ and

$$f(r, s)f(rs, t) = f(r, st)f(s, t)$$
for all \( r, s, t \in T \). We denote by \( \mathcal{F}(T, E) \) the set of Schur \( E \)-functions for \( T \) and put

\[
\begin{align*}
\tilde{f} & : T \rightarrow Un E^c, \quad t \mapsto f(t, t^{-1})^*, \\
\hat{f} & : T \times T \rightarrow Un E^c, \quad (s, t) \mapsto f(t^{-1}, s^{-1})
\end{align*}
\]

for every \( f \in \mathcal{F}(T, E) \).

Schur functions are also called normalized factor set or multiplier or two-co-cycle (for \( T \) with values in \( Un E^c \)) in the literature. We present in this section only some elementary properties (which will be used in the sequel) in order to fix the notation and the terminology. By the way, \( Un E^c \) can be replaced in this section by an arbitrary commutative multiplicative group (with \( * \) replaced by \( -1 \)).

**PROPOSITION 1.1.2** Let \( f \in \mathcal{F}(T, E) \).

a) For every \( t \in T \),

\[
f(t, 1) = f(1, t) = 1_G, \quad f(t, t^{-1}) = f(t^{-1}, t), \quad \tilde{f}(t) = \tilde{f}(t^{-1}).
\]

b) For all \( s, t \in T \),

\[
f(s, t)\tilde{f}(s) = f(s^{-1}, st)^*, \quad f(s, t)\hat{f}(t) = f(st, t^{-1})^*.
\]

a) Putting \( s = 1 \) in the equation of \( f \) we obtain

\[
f(r, 1)f(r, t) = f(r, t)f(1, t)
\]

so

\[
f(r, 1) = f(1, t)
\]

for all \( r, t \in T \). Hence

\[
f(t, 1) = f(1, t) = f(1, 1) = 1_G.
\]

Putting \( r = t \) and \( s = t^{-1} \) in the equation of \( f \) we get

\[
f(t, t^{-1})f(1, t) = f(t, 1)f(t^{-1}, t).
\]

By the above,

\[
f(t, t^{-1}) = f(t^{-1}, t), \quad \tilde{f}(t) = \tilde{f}(t^{-1}).
\]

b) Putting \( r = s^{-1} \) in the equation of \( f \), by a),

\[
f(s, t)f(s^{-1}, st) = f(s^{-1}, s)f(1, t) = \tilde{f}(s)^*, \\
\tilde{f}(s) = f(s^{-1}, st)^*.
\]

Putting now \( t = s^{-1} \) in the equation of \( f \), by a) again,

\[
f(r, s)f(rs, s^{-1}) = f(r, 1)f(s, s^{-1}) = \tilde{f}(s)^*, \\
f(r, s)\tilde{f}(s) = f(rs, s^{-1})^*, \quad f(s, t)\hat{f}(t) = f(st, t^{-1})^*.
\]

**DEFINITION 1.1.3** We put

\[
\Lambda(T, E) := \{ \lambda : T \rightarrow Un E^c | \lambda(1) = 1_G \}
\]

and

\[
\hat{\lambda} : T \rightarrow Un E^c, \quad t \mapsto \lambda(t^{-1}), \\
\tilde{\lambda} : T \times T \rightarrow Un E^c, \quad (s, t) \mapsto \lambda(s)\lambda(t)\lambda(st)^*
\]

for every \( \lambda \in \Lambda(T, E) \).
PROPOSITION 1.1.4

a) $\mathcal{F}(T, E)$ is a subgroup of the commutative multiplicative group $(\text{Un } E)^T$ such that $f^*$ is the inverse of $f$ for every $f \in \mathcal{F}(T, E)$.
b) $f^* \in \mathcal{F}(T, E)$ for every $f \in \mathcal{F}(T, E)$ and the map

$$\mathcal{F}(T, E) \longrightarrow \mathcal{F}(T, E), \quad f \longmapsto \hat{f}$$

is an involutive group automorphism.
c) $\Lambda(T, E)$ is a subgroup of the commutative multiplicative group $(\text{Un } E)^T$, $\delta \lambda \in \mathcal{F}(T, E)$ for every $\lambda \in \Lambda(T, E)$, and the map

$$\delta : \Lambda(T, E) \longrightarrow \mathcal{F}(T, E), \quad \lambda \longmapsto \delta \lambda$$

is a group homomorphism with kernel

$$\{\lambda \in \Lambda(T, E) \mid \lambda \text{ is a group homomorphism} \}$$

such that $\hat{\delta \lambda} = \delta \hat{\lambda}$ for every $\lambda \in \Lambda(T, E)$.

a) is obvious.
b) For $r, s, t \in T$,

$$\hat{f}(r, s)\hat{f}(rs, t) = f(s^1, r^{-1})f(t^{-1}, s^1r^{-1}) = f(t^{-1}, s^1)f(t^{-1}s^{-1}, r^{-1}) = \hat{f}(r, st)\hat{f}(s, t),$$

so $\hat{f} \in \mathcal{F}(T, E)$. For $f, g \in \mathcal{F}(T, E)$,

$$\hat{fg}(s, t) = (fg)(t^{-1}, s^1) = f(t^{-1}, s^1)g(t^{-1}, s^1) = \hat{f}(s, t)\hat{g}(s, t) = (\hat{fg})(s, t),$$

$$\hat{f}^*(s, t) = \hat{f}(s, t)^* = f(t^{-1}, s^1)^* = f^*(t^{-1}, s^1) = \hat{f}^*(s, t), \quad (\hat{f}^*)^* = \hat{f}^*.$$  

c) For $r, s, t \in T$,

$$\delta \lambda(r, s)\delta \lambda(rs, t) = \lambda(r)\lambda(s)\lambda(rs)\lambda(rs)\lambda(t)\lambda(rst)^* = \lambda(r)\lambda(s)\lambda(t)\lambda(rst)^*,$$

$$\delta \lambda(r, st)\delta \lambda(s, t) = \lambda(r)\lambda(st)\lambda(rst)\lambda(s)\lambda(t)\lambda(st)^* = \lambda(r)\lambda(s)\lambda(t)\lambda(rst)^*$$

so $\delta \lambda \in \mathcal{F}(T, E)$. For $\lambda, \mu \in \mathcal{F}(T, E)$ and $s, t \in T$,

$$\delta \lambda(s, t)\delta \mu(s, t) = \lambda(s)\lambda(t)\lambda(st)^*\mu(s)\mu(t)\mu(st)^* = (\delta \lambda(s)\delta \mu(t)\delta \mu(st)^*) = \delta \lambda \delta \mu(s, t),$$

$$\delta \lambda^*(s, t) = \lambda^*(s)\lambda^*(t)\lambda(st) = (\delta \lambda(s, t))^* = (\delta \lambda)^*(s, t), \quad \delta \lambda^* = (\delta \lambda)^*,$$

so $\delta$ is a group homomorphism. The other assertions are obvious.

PROPOSITION 1.1.5 Let $t \in T$, $m, n \in \mathbb{Z}$, and $f \in \mathcal{F}(T, E)$.

a) $f(t^m, t^n) = f(t^m, t^n)$.
b) $m \in \mathbb{N} \Rightarrow f(t^m, t^n) = \left(\prod_{i=0}^{m-1} f(t^{i+1}, t)\right)^{n-1} \prod_{k=1}^{n-1} f(t^k, t)$.
c) We define

$$\lambda : \mathbb{Z} \longrightarrow \text{Un } E^\circ, \quad n \longmapsto \left\{ \begin{array}{ll} \prod_{i=1}^{n-1} f(t^i, t)^*, & \text{if } n \in \mathbb{N} \\ \prod_{i=1}^{n-1} f(t^i, t) & \text{if } n \not\in \mathbb{N} \end{array} \right.$$  

If $t^p \neq 1$ for every $p \in \mathbb{N}$ then

$$f(t^m, t^n) = \lambda(m)\lambda(n)\lambda(m+n)^*$$

for all $m, n \in \mathbb{Z}$.
a) We may assume $m \in \mathbb{N}$ because otherwise we can replace $t$ by $t^{-1}$. Put

$$P(m, n) : \Leftrightarrow f(t^m, t^n) = f(t^n, t^m),$$

$$Q(m) : \Leftrightarrow P(m, n) \text{ holds for all } n \in \mathbb{Z}.$$  

From

$$f(t^m, t^{n+m}) f(t^n, t^m) = f(t^m, t^n) f(t^{n+m}, t^m)$$

it follows

$$P(m, n) \Leftrightarrow P(m, n-m) \Leftrightarrow P(m, n-km)$$

for all $k \in \mathbb{Z}$.

We prove the assertion by induction. $P(m, 0)$ follows from Proposition 1.1.2 a). By the above

$$P(1, 0) \Leftrightarrow P(1, k)$$

for all $k \in \mathbb{Z}$. Thus $Q(1)$ holds.

Assume $Q(p)$ holds for all $p \in \mathbb{N}_{m-1}$. Then $P(m, p)$ holds for all $p \in \mathbb{N}_{m-1} \cup \{0\}$. Let $n \in \mathbb{Z}$. There is a $k \in \mathbb{Z}$ such that

$$p := n-km \in \mathbb{N}_{m-1} \cup \{0\}.$$  

By the above $P(m, n)$ holds. Thus $Q(m)$ holds and this finishes the inductive proof.

b) We prove the formula by induction with respect to $m$. By a), the formula holds for $m = 1$. Assume the formula holds for an $m \in \mathbb{N}$. Since

$$f(t^m, t) f(t^{m+1}, t^0) = f(t^m, t^{m+1}) f(t^0, t)$$

we get by a),

$$f(t^{m+1}, t^0) = f(t^m, t^{m+1}) f(t^0, t) f(t^m, t)^* =$$

$$= \left( \prod_{j=0}^{m-1} f(t^{n+1+j}, t) \right) \left( \prod_{k=1}^{m-1} f(t^k, t)^* \right) f(t^0, t) f(t^m, t)^* =$$

$$= \left( \prod_{j=0}^{m} f(t^{m+j}, t) \right) \left( \prod_{k=1}^{m} f(t^k, t)^* \right).$$

Thus the formula holds also for $m + 1$.

c) If $m, n \in \mathbb{N}$ then by b),

$$\lambda(m) \lambda(n) \lambda(m+n)^* =$$

$$= \left( \prod_{k=1}^{m-1} f(t^k, t)^* \right) \left( \prod_{j=1}^{n-1} f(t^j, t)^* \right) \left( \prod_{j=1}^{m+n-1} f(t^j, t) \right) =$$

$$= \left( \prod_{j=0}^{m-1} f(t^{n+j}, t) \right) \left( \prod_{k=1}^{m} f(t^k, t)^* \right) = f(t^m, t^n).$$
If \( m, n \in \mathbb{N} \), \( n \leq m-1 \) then by b),

\[
\lambda(m)\lambda(-n)\lambda(m-n)^* = \\
= \left( \prod_{j=1}^{m-1} f(t^j, t^*) \right) \left( \prod_{j=1}^n f(t^j, t) \right) \left( \prod_{j=1}^{m-n-1} f(t^j, t) \right) = \\
= \left( \prod_{j=0}^{m-1} f(t^{-nj}, t) \right) \left( \prod_{k=1}^{m-n} f(t^k, t^*) \right) = f(t^m, t^n).
\]

If \( m, n \in \mathbb{N} \), \( n \geq m \) then by b),

\[
\lambda(m)\lambda(-n)\lambda(m-n)^* = \\
= \left( \prod_{k=1}^{m-1} f(t^k, t^*) \right) \left( \prod_{j=1}^n f(t^j, t) \right) \left( \prod_{j=1}^{m-n} f(t^j, t^*) \right) = \\
= \left( \prod_{j=-n+1}^{n-1} f(t^j, t) \right) \left( \prod_{k=1}^{m-n} f(t^k, t^*) \right) = f(t^m, t^n).
\]

For all \( m, n \in \mathbb{N} \) put

\[ R(m, n) : \iff f(t^m, t^n) = \lambda(-m)\lambda(-n)\lambda(-m-n)^*. \]

By the above and by Proposition 1.1.2 a),b),

\[
\lambda(-1)\lambda(-1)\lambda(-2)^* = f(t^{-1}, t)f(t^{-2}, t)^* = \overline{f(t^{-1})^*f(t, t^{-2})^*} = f(t^3, t^{-1}),
\]

so \( R(1, 1) \) holds. Let now \( m, n \in \mathbb{N} \) and assume \( R(m, n) \) holds. Then

\[
\lambda(-m)\lambda(-n-1)\lambda(-m-n-1)^* = \\
= \left( \prod_{j=1}^{m} f(t^j, t) \right) \left( \prod_{j=1}^{n+1} f(t^j, t) \right) \left( \prod_{j=1}^{m+n+1} f(t^j, t^*) \right) = \\
= f(t^{-m-n^*} f(t^{m-n-1}, t) f(t^{m-n-1}, t)^*) = f(t^{m-n^*} f(t^{m-n-1}, t), t^*).
\]

so \( R(m, n) \Rightarrow R(m, n + 1) \). By symmetry and a), \( R(m, n) \) holds for all \( m, n \in \mathbb{N} \).

\[ \blacksquare \]

**COROLLARY 1.1.6** The map

\[ \Lambda(\mathbb{Z}, E) \longrightarrow \mathcal{F}(\mathbb{Z}, E), \quad \lambda \mapsto \delta \lambda \]

is a surjective group homomorphism with kernel

\[ \{ \lambda \in \Lambda(\mathbb{Z}, E) \mid n \in \mathbb{Z} \Rightarrow \lambda(n) = \lambda(1)^n \}. \]

By Proposition 1.1.4 c), only the surjectivity of the above map has to be proved and this follows from Proposition 1.1.5 c).

\[ \blacksquare \]

### 1.2 \( E \)-C*-algebras

By replacing the scalars with the unital \( C^* \)-algebra \( E \) we restrict the category of \( C^* \)-algebras to the subcategory of those \( C^* \)-algebras which are connected in a certain way with \( E \). The category of unital \( C^* \)-algebras is replaced by the category of \( E \)-C*-algebras, while the general category of \( C^* \)-algebras is replaced by the category of adapted \( E \)-modules. The Schur products are always adapted \( E \)-modules.
**DEFINITION 1.2.1** We call in this article an **E-module** a C*-algebra $F$ endowed with the bilinear maps

$$
E \times F \rightarrow F, \quad (\alpha, x) \rightarrow \alpha x,
$$

$$
F \times E \rightarrow F, \quad (x, \alpha) \rightarrow x\alpha
$$
such that for all $\alpha, \beta \in E$ and $x, y \in F$,

$$
(\alpha \beta)x = \alpha(\beta x), \quad \alpha(x\beta) = (\alpha x)\beta, \quad x(\alpha \beta) = (x\alpha)\beta,
$$

$$
\alpha(xy) = (\alpha x)y, \quad (xy)\alpha = x(y\alpha), \quad \alpha \in E^* \Rightarrow \alpha x = x\alpha,
$$

$$
(x\alpha)^* = x^*\alpha^*, \quad (\alpha x)^* = \alpha^*x^*.
$$

If $F, G$ are E-modules then a C*-homomorphism $\varphi : F \rightarrow G$ is called **E-linear** if for all $(\alpha, x) \in E \times F$,

$$
\varphi(\alpha x) = \alpha(\varphi x), \quad \varphi(x\alpha) = (\varphi x)\alpha.
$$

For all $(\alpha, x) \in E \times F$,

$$
\|\alpha x\|^2 = \|x^*\alpha^*x\| \leq \|x\|^2\|\alpha\|^2, \quad \|x\alpha\|^2 = \|\alpha^*x^*x\| \leq \|\alpha\|^2\|x\|^2
$$

so

$$
\|\alpha x\| \leq \|\alpha\||\|x\|, \quad \|x\alpha\| \leq \|\alpha\||\|x\|\|\alpha\|.
$$

**DEFINITION 1.2.2** An **E-C**-**algebra** is a unital C**-**algebra $F$ for which $E$ is a canonical unital C**-**subalgebra such that $E^*$ defined with respect to $E$ coincides with $F^*$ defined with respect to $F$ i.e. for every $x \in E$, if $xy = yx$ for all $y \in E$ then $xy = yx$ for all $y \in F$. Every closed ideal of an E-C**-**algebra is canonically an E-module.

Let $F, G$ be E-C**-**algebras. A map $\varphi : F \rightarrow G$ is called an **E-C****-****-homomorphism** if it is an E-linear C**-**homomorphism. If in addition $\varphi$ is a C**-**isomorphism then we say that $\varphi$ is an **E-C****-**isomorphism and we use in this case the notation $\simeq_E$. A C**-**subalgebra $F_0$ of $F$ is called **E-C****-****-subalgebra** of $F$ if $E \subset F_0$.

With the notation of the above Definition $(\alpha - \varphi \alpha)\varphi x = 0$ for all $\alpha \in E$ and $x \in F$. Thus $\varphi$ is unital iff $\varphi\alpha = \alpha$ for every $\alpha \in E$. The example

$$
\mathbb{K} \rightarrow \mathbb{K} \times \mathbb{K}, \quad x \mapsto (x, 0)
$$

shows that an E-C**-**homomorphism need not be unital.

If we put $T = \{z \in \mathbb{C} \mid |z| = 1\}, E = \mathcal{C}(T, \mathbb{C})$, and

$$
x : T \rightarrow \mathbb{C}, \quad z \mapsto z
$$

and if we denote by $\lambda$ the Lebesgue measure on $T$ then $L^\infty(\lambda)$ is an E-C**-**algebra, $x \in Un E$, and $x$ is homotopic to $1_E$ in Un $L^\infty(\lambda)$ but not in Un $\mathcal{C}(T, \mathbb{C})$.

**DEFINITION 1.2.3** We denote by $\mathcal{E}_E$ (resp. by $\mathcal{P}_E^1$) the category of E-C**-**algebras for which the morphisms are the E-C**-**homomorphisms (resp. the unital E-C**-**homomorphisms).

**PROPOSITION 1.2.4** Let $F$ be an E-module.

a) We denote by $\hat{F}$ the vector space $E \times F$ endowed with the bilinear map

$$
(E \times F) \times (E \times F) \rightarrow E \times F, \quad ((\alpha, x), (\beta, y)) \mapsto (\alpha\beta, ay + x\beta + xy)
$$

and with the conjugate linear map

$$
E \times F \rightarrow E \times F, \quad (\alpha, x) \mapsto (\alpha^*, x^*).
$$

$\hat{F}$ is an involutive unital algebra with $(1_E, 0)$ as unit.

b) The maps
are involutive algebra homomorphisms such that \( \pi \circ \lambda \) is the identity map of \( E \), \( \lambda \) and \( \iota \) are injective, and \( \lambda \) and \( \pi \) are unital. If there is a norm on \( \hat{F} \) with respect to which it is a C*-algebra (in which case such a norm is unique), then we call \( F \) adapted. We denote by \( \mathcal{M}_E \) the category of adapted \( E \)-modules for which the morphism are the \( E \)-linear C*-homomorphisms.

c) If \( F \) is adapted then \( \hat{F} \) is an \( E \)-C*-algebra by using canonically the injection \( \lambda \) and for all \( \alpha \in E \) and \( x \in F \),

\[
\|\alpha\| = \|\alpha, x\| = \|\alpha\| + \|x\|, \quad \|0, x\| = \|x\| \leq 2\|\alpha, x\|.
\]

In particular \( F \) (identified with \( \iota(F) \)) is a closed ideal of \( \hat{F} \).

d) If \( E \) and \( F \) are C*-subalgebras of a C*-algebra \( G \) in such a way that the structure of \( E \)-module of \( F \) is inherited from \( G \) then

\[
\varphi : \hat{F} \to E \times G, \quad (\alpha, x) \mapsto (\alpha, \alpha + x)
\]

is an injective involutive algebra homomorphism, \( \varphi(\hat{F}) \) is closed, \( F \) is adapted, and for all \( \alpha \in E \) and \( x \in F \),

\[
\|\alpha, x\| = \sup \{\|\alpha\|, \|\alpha + x\|\}.
\]

In particular every closed ideal of an \( E \)-C*-algebra is adapted and \( \mathcal{C}_E \) is a full subcategory of \( \mathcal{M}_E \).

e) A closed ideal \( G \) of an adapted \( E \)-module \( F \), which is at the same time an \( E \)-submodule of \( F \), is adapted.

f) If \( F \) is unital then it is adapted and

\[
\hat{F} \to \mathbb{R}_+, \quad (\alpha, x) \mapsto \sup \{\|\alpha\|, \|\alpha F + x\|\}
\]

is the \( C^* \)-norm of \( F \).

g) If

\[
\lim_{y \to \hat{F}} \|\alpha y - \alpha x\| = 0
\]

for all \( \alpha \in E_n \) where \( \hat{F} \) denotes the canonical approximate unit of \( F \), then \( F \) is adapted and

\[
\hat{F} \to \mathbb{R}_+, \quad (\alpha, x) \mapsto \sup \{\|\alpha\|, \limsup_{y \to \hat{F}} \|\alpha y + x\|\}
\]

is the \( C^* \)-norm of \( F \). In particular \( F \) is adapted if \( E \) is commutative.

h) If \( F \) is an adapted \( E \)-module then (with the notation of b))

\[
0 \to F \to \hat{F} \to \mathcal{M}_E \to 0
\]

is a split exact sequence in the category \( \mathcal{M}_E \).

a) and b) are easy to see.

c) Since \( \lambda \) and \( \iota \) are injective and,

\[
\pi(\alpha, x) = \alpha, \quad (\alpha, x) = (\alpha, 0) + (0, x),
\]

\[
(\alpha, 0)(0, x) = (0, \alpha x), \quad (0, x)(\alpha, 0) = (0, x\alpha)
\]
we get the first and the last two inequalities as well as the identity $(0, x) = \|x\|$. It follows
\[
\|(0, x)\| \leq \|\varphi(x)\| + \|\varphi(0)\| = \|\alpha, x\| + \|\lambda\alpha(\alpha, x)\| \leq \\
\leq \|\alpha, x\| + \|\alpha, x\| = 2\|\alpha, x\|.
\]
d) It is easy to see that $\varphi$ is an injective involutive algebra homomorphism. Let $(\alpha, x) \in \varphi(\tilde{F})$. There are sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ in $E$ and $F$, respectively, such that
\[
\lim_{n \to \infty} (\alpha_n, \alpha_n + x_n) = (\alpha, x).
\]
it follows
\[
\alpha = \lim_{n \to \infty} \alpha_n \in E, \quad x - \alpha = \lim_{n \to \infty} x_n \in F, \quad (\alpha, x) = \varphi(x - \alpha) \in \varphi(\tilde{F}).
\]
Thus $\varphi(\tilde{F})$ is closed, which proves the assertion by pulling back the norm of $E \times G$.
e) By c), $F$ is a closed ideal of $\tilde{F}$ so $\tilde{G}$ is a closed ideal of $\tilde{F}$ (use an approximate unit of $F$). Since $\tilde{G}$ is an $E$-module of $\tilde{F}$ its structure of $E$-module is inherited from $F$. By d), $\tilde{G}$ is adapted.
f) The map
\[
\tilde{F} \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha, \alpha1_{\tilde{F}} + x)
\]
is an isomorphism of involutive algebras and so we can pull back the norm of $E \times F$.
g) It is easy to see that the above map is a norm. Since
\[
\sup\{\|\alpha\|, \frac{1}{2}\|x\|\} \leq \|(\alpha, x)\| \leq \|\alpha\| + \|x\|
\]
for all $(\alpha, x) \in E \times F$, $\tilde{F}$ endowed with this norm is complete. For $(\alpha, x) \in E \times F$,
\[
(\alpha, x)^*(\alpha, x) = (\alpha^*\alpha, \alpha^*x + x^*\alpha + x^*x),
\]
\[
\|(\alpha, x)^*(\alpha, x)\| = \sup\{\|\alpha\|^2, \lim_{y, \delta} \sup \|\alpha^*\alpha' + \alpha^*x + x^*\alpha + x^*x\|\}.
\]
For $y \in F^+_\tilde{F}$,
\[
\|(\alpha^2/x + x)^*(\alpha^2/x + x) - (\alpha^*\alpha'y + \alpha^*x + x^*\alpha + x^*x)\| \leq \\
\leq \|\lambda^2\alpha^*\alpha - \alpha^*\alpha'y\| + \|\lambda^2\alpha^*x - x^*\alpha'y\| + \|\lambda^2\alpha^*\alpha - \alpha^*\alpha'y\| + \|\lambda^2\alpha^*x - x^*\alpha'y\|
\]
so
\[
\lim_{y, \delta} \|(\alpha^2/x + x)^*(\alpha^2/x + x) - (\alpha^*\alpha'y + \alpha^*x + x^*\alpha + x^*x)\| = 0.
\]
Since the map $F_+ \longrightarrow F_+, y \longmapsto \tilde{y}^2$ maps $\tilde{F}$ into itself and
\[
\|\alpha y + x\|^2 = \|\alpha y + x\|^2 = \|\alpha y + x\|^2
\]

Thus the above norm is a C*-norm and \( F \) is adapted.

**COROLLARY 1.2.5** Let \( F \) an \( E \)-module, \( G \) a C*-algebra, and \( \otimes _{\alpha } \) the spatial tensor product.

a) \( F \otimes _{\alpha } G \) is in a natural way an \( E \)-module the multiplication being given by
\[
\alpha (x \otimes y) = (ax) \otimes y, \quad (x \otimes y)\alpha = (x\alpha) \otimes y
\]
for all \( \alpha \in E, x \in F, \) and \( y \in G \).
b) If \( F \) is an \( E \)-C*-algebra and \( G \) is unital then the map
\[
E \longrightarrow F \otimes _{\alpha } G, \quad \alpha \mapsto \alpha \otimes 1_{C}
\]
is an injective C*-homomorphism. In particular, the \( E \)-module \( F \otimes _{\alpha } G \) is an \( E \)-C*-algebra.
c) If \( F \) is an adapted \( E \)-module then the \( E \)-module \( F \otimes _{\alpha } G \) is adapted and
\[
\| (\alpha, z) \| = \sup \{ \| \alpha \|, \| \alpha + z \| \}
\]
for all \( (\alpha, z) \in E \times (F \otimes _{\alpha } G) \).
d) If \( F \) is an adapted \( E \)-module and \( G = C_{0}(\Omega) \) for a locally compact space \( \Omega \) then \( C_{0}(\Omega, F) \) is adapted and
\[
\| (\alpha, x) \| = \sup \{ \| \alpha \|, \| \alpha x + x \| \}
\]
for all \( (\alpha, x) \in E \times C_{0}(\Omega, F) \).

a) and b) are easy to see.
c) If \( \tilde{G} \) denotes the unitization of \( G \) then by b), \( \tilde{F} \otimes _{\alpha } \tilde{G} \) is an \( E \)-C*-algebra and \( F \otimes _{\alpha } G \) is a closed ideal of it, so the assertion follows from Proposition 1.2.4 d), e).
d) follows from c).

**PROPOSITION 1.2.6**

a) If \( F, G \) are \( E \)-modules and \( \varphi : F \to G \) is an \( E \)-linear C*-homomorphism then the map
\[
\tilde{\varphi} : \tilde{F} \longrightarrow \tilde{G}, \quad (\alpha, x) \longmapsto (\alpha, \varphi x)
\]
is an involutive unital algebra homomorphism, injective or surjective if \( \varphi \) is so. If \( F = G \) and if \( \varphi \) is the identity map then \( \tilde{\varphi} \) is also the identity map.
b) Let \( F_1, F_2, F_3 \) be \( E \)-modules and let \( \varphi : F_1 \to F_2 \) and \( \psi : F_2 \to F_3 \) be \( E \)-linear C*-homomorphisms. Then \( \tilde{\psi} \circ \tilde{\varphi} = \tilde{\psi} \circ \tilde{\varphi} \). 

**PROPOSITION 1.2.7** Let \( G \) be an \( E \)-module, \( F \) an \( E \)-submodule of \( G \) which is at the same time an ideal of \( G \), and \( \varphi : G \to G/F \) the quotient map.

a) \( G/F \) has a natural structure of \( E \)-module and \( \varphi \) is \( E \)-linear,
b) If \( G \) is adapted then \( G/F \) is also adapted. Moreover if \( \psi : \tilde{G} \longrightarrow \tilde{G}/\tilde{F} \) denotes the quotient map (where \( F \) is identified to \( \{(0, x) \mid x \in F\} \)) then there is an \( E \)-C*-isomorphism \( \theta : \tilde{G}/\tilde{F} \longrightarrow \tilde{G}/\tilde{F} \) such that \( \psi = \theta \circ \tilde{\varphi} \).
a) is easy to see.
b) Let \((\alpha, x) \in \tilde{F}/\tilde{F}\) and let \(x, y \in \varphi^{-1}(z)\). Then \(\varphi(\alpha, x) = \varphi(\alpha, y)\) and we put \(\theta(\alpha, z) := \varphi(\alpha, x)\). It is straightforward to show that \(\theta\) is an isomorphism of involutive algebras. By pulling back the norm of \(\tilde{G}/F\) with respect to \(\theta\) we see that \(G/F\) is adapted. \(\square\)

**LEMMA 1.2.8** Let \(\{(\tilde{F}_i), (\varphi_i)_{i \in I}\}\) be an inductive system in the category of \(C^*\)-algebras, \((F_i, (\varphi_i)_{i \in I})\) its inductive limit, \(G\) a \(C^*\)-algebra, for every \(i \in I\), \(\varphi_i : F_i \rightarrow G\) a \(C^*\)-homomorphism such that \(\varphi_i \circ \varphi_j = \varphi_i\) for all \(i, j \in I\), \(i \leq j\), and \(\psi : F \rightarrow G\) the resulting \(C^*\)-homomorphism. If \(\ker \varphi_i \subseteq \ker \varphi_i\) for every \(i \in I\) then \(\psi\) is injective.

Let \(i \in I\). Since \(\ker \varphi_i \subseteq \ker \varphi_i\) is obvious, we have \(\ker \varphi_i = \ker \varphi_i\). Let \(p : F_i \rightarrow F_i/\ker \varphi_i\) be the quotient map and
\[
\varphi'_i : F_i/\ker \varphi_i \rightarrow F_i, \quad \psi'_i : F_i/\ker \varphi_i \rightarrow G
\]
the injective \(C^*\)-homomorphisms with
\[
\varphi_i = \varphi'_i \varphi, \quad \psi_i = \psi'_i \varphi.
\]
Then
\[
\psi \varphi = \psi_i = \varphi \varphi_i = \varphi_i \varphi \varphi_i.
\]

For \(x \in F_i\), since \(\varphi_i'\) and \(\varphi_i\) are norm-preserving,
\[
\|\varphi x\| = \|\varphi'_i \varphi x\| = \|\varphi_i \varphi x\| \leq \|\varphi'_i \varphi x\| = \|\varphi x\|,
\]
\[
\|\varphi \varphi_i x\| = \|\varphi_i \varphi x\| = \|\varphi_i \varphi x\| = \|\varphi x\|.
\]

Thus \(\psi\) preserves the norms on \(\cup_{i \in I} F_i\). Since this set is dense in \(F\), \(\psi\) is injective.

**PROPOSITION 1.2.9** Let \(\{(\tilde{F}_i), (\varphi_i)_{i \in I}\}\) be an inductive system in the category \(\mathfrak{M}_E\) and let \((F_i, (\varphi_i)_{i \in I})\) be its inductive limit in the category of \(E\)-modules (Proposition 1.2.4 c)).

a) \(F\) is adapted.

b) Let \((G, (\psi_i)_{i \in I})\) be the inductive limit in the category \(\mathfrak{C}_E\) of the inductive system \(\{(\tilde{F}_i), (\varphi_i)_{i \in I}\}\) (Proposition 1.2.6 a),b)) and let \(\psi : G \rightarrow \tilde{F}\) be the unital \(C^*\)-homomorphism such that \(\psi \varphi_i = \tilde{\varphi}\) for every \(i \in I\). Then \(\psi\) is an \(E\)-\(C^*\)-isomorphism.

a) Put
\[
F_0 := \{(\alpha, x) \in \tilde{F} \mid \alpha \in E, \ x \in \bigcup_{i \in I} \varphi_i(F_i)\},
\]
\[
p : F_0 \rightarrow \mathbb{R}_+, \quad (\alpha, x) \mapsto \inf\{\|x_i\| \mid i \in I, x_i \in F_i, \varphi_i x_i = x\}.
\]

\(F_0\) is an involutive unital subalgebra of \(\tilde{F}\). \(p\) is a norm and by Proposition 1.2.4 c),
\[
q(\alpha, x) := \lim_{\substack{\alpha, y \in F_0 \\ y \rightarrow x}} p(\alpha, y)
\]
exists and
\[
\|\alpha\| \leq q(\alpha, x) \leq \|\alpha\| + \|x\|, \quad \|x\| \leq 2q(\alpha, x)
\]
for every \((\alpha, x) \in \tilde{F}\).

Let \((\alpha, x) \in F_0\). Let further \(i \in I\), \(x_i, y_i \in F_i\) with \(\varphi_i x_i = x, \varphi_i y_i = \alpha^* x + x^* \alpha + x^* x\). Then
\[
(0, \varphi_i(\alpha^* x_i + x_i^* \alpha + x_i^* x_i - y_i)) = \tilde{\varphi}_i((\alpha, x_i)^*(\alpha, x_i) - (\alpha^* \alpha, y_i)) = 0
\]
so
\[
\inf_{i \leq j} \|\varphi_j(\alpha^* x_i + x_i^* \alpha + x_i^* x_i - y_i)\| = 0.
\]
For $\epsilon > 0$ there is a $j \in I$, $i \leq j$, with
\[
\|\varphi_j(\alpha^* x_i + x_i^* x + x_i^* x_i - y_i)\| < \epsilon.
\]
We get
\[
p(\alpha, x)^2 \leq \|\varphi_j(\alpha, x)\|^2 = \|\varphi_j(\alpha, x)^*(\alpha, x)\| = \\
= \|\varphi_j(\alpha^* x_i + x_i^* x + y_i)\|^2 = \\
= \|\varphi_j(\alpha^* x_i + x_i^* x + x_i^* x_i - y_i)\| \leq \\
\leq \|\varphi_j(\alpha, x)\| + \|\varphi_j(\alpha^* x_i + x_i^* x + x_i^* x_i - y_i)\| < \|\varphi_j(\alpha, x)\| + \epsilon.
\]

By taking the infimum on the right side it follows, since $\epsilon$ is arbitrary,
\[
p(\alpha, x)^2 \leq p(\alpha^* x_i + x^* x + x^* x) = p((\alpha, x)^*(\alpha, x))
\]

and this shows that $p$ is a C*-norm. It is easy to see that $q$ is a C*-norms. By the above inequalities, $\tilde{F}$ endowed with the norm $q$ is complete, i.e. $\tilde{F}$ is a C*-algebra and $F$ is adapted.

b) Let $i \in I$ and let $(\alpha, x) \in \text{Ker } \tilde{\varphi}_j$. Then
\[
0 = \tilde{\varphi}_j(\alpha, x) = (\alpha, \varphi_j x)
\]
so
\[
\alpha = 0, \quad \varphi_j x = 0, \quad \inf_{j \in I_{ij}} \|\varphi_j x\| = 0,
\]
\[
\|\tilde{\varphi}_j(0, x)\| = \|(0, \varphi_j x)\| = \|\varphi_j x\|,
\]
\[
\|\psi_j(\alpha, x)\| = \inf_{j \in I_{ij}} \|\tilde{\varphi}_j(0, x)\| = 0, \quad (\alpha, x) \in \text{Ker } \psi_j.
\]

By Lemma 1.2.8, $\psi$ is injective.

Let $(\beta, y) \in \tilde{F}$ and let $\epsilon > 0$. There are $i \in I$ and $x \in F_i$ with $\|\varphi_j x - y\| < \epsilon$. Then
\[
\psi \psi_j(\beta, y) = \tilde{\varphi}_j(\beta, x) = (\beta, \varphi_j x).
\]
\[
\|\psi \psi_j(\beta, y) - (\beta, y)\| = \|\tilde{\varphi}_j(\beta, x) - (\beta, y)\| = \|\varphi_j x - y\| < \epsilon.
\]

Thus $\psi(G)$ is dense in $\tilde{F}$ and $\psi$ is surjective. Hence $\psi$ is a C*-isomorphism.

**COROLLARY 1.2.10** We put $\Phi_E(F) := \tilde{F}$ for every E-module $F$ and similarly $\Phi_E(\varphi) := \tilde{\varphi}$ for every E-linear C*-homomorphism $\varphi$.

a) $\Phi_E$ is a covariant functor from the category $\mathfrak{M}_E$ in the category $\mathfrak{C}^*_E$.

b) The categories $\mathfrak{C}^*_E$ and $\mathfrak{M}_E$ possess inductive limits and the functor $\Phi_E$ is continuous with respect to the inductive limits.

a) follows from Proposition 1.2.6.

b) follows from Proposition 1.2.9.

**Remark**. The category $\mathfrak{C}^*_E$ does not possess inductive limits in general. This happens for instance if $\varphi_j = 0$ for all $i, j \in I$.

### 1.3 Some topologies

The purely algebraic projective representation of a group produces only an involutive algebra. In order to obtain a C*-algebra we need to take the closure with respect to a certain topology. For this purpose we shall define some different topologies, but it will be shown that all these topologies conduct to the same construction. The use of more topologies simplifies the manipulations.
We introduce the following notation in order to unify the cases of C*-algebras and (resp. W*-algebras).

**DEFINITION 1.3.1**

\[ \oplus := \oplus \quad \text{(resp. } \oplus := \oplus \text{)}, \]
\[ \otimes := \otimes \quad \text{(resp. } \otimes := \otimes \text{)}, \]
\[ \sum := \sum \quad \text{(resp. } \sum := \sum \text{)}. \]

If \( \tau \) is a Hausdorff topology on \( L_{\mathcal{E}}(H) \) then for every \( G \subset L_{\mathcal{E}}(H) \), \( G_\tau \) denotes the set \( G \) endowed with the relative topology \( \tau \) and \( \overline{G}^\tau \) denotes the closure of \( G \) in \( L_{\mathcal{E}}(H) \). Moreover \( \sum^\tau \) denotes the sum with respect to \( \tau \).

**LEMMA 1.3.2** For \( x \in E \), by the above identification of \( E \) with \( L_{\mathcal{E}}(\tilde{E}) \),

\[ x \otimes 1_K : H \to H, \quad \xi \mapsto (x \xi_t)_{t \in T} \]

is well-defined and belongs to \( L_{\mathcal{E}}(H) \).

a) The map

\[ \varphi : E \to L_{\mathcal{E}}(H), \quad x \mapsto x \otimes 1_K \]

is an injective unital C*-homomorphism.

b) Assume \( E \) is a W*-algebra. Then for every \( (a, \xi, \eta) \in E \times H \times H \), the family \( (\xi, a_n^\tau)_{t \in T} \) is summable in \( \tilde{E} \) and for every \( x \in E \),

\[ \langle \varphi x, (a, \xi, n) \rangle = \left( x, \sum_{t \in T} \xi_t a_n^\tau \right). \]

Thus \( \varphi \) is a W*-homomorphism (\([1] \) Theorem 5.6.3.5 d)) with

\[ \widetilde{\varphi}(a, \xi, \eta) = \sum_{t \in T} \xi_t a_n^\tau, \]

where \( \widetilde{\varphi} \) denotes the pretranspose of \( \varphi \).

c) If we consider \( E \) as a canonical unital C**-subalgebra of \( L_{\mathcal{E}}(H) \) by using the embedding of a) then \( L_{\mathcal{E}}(H) \) is an \( E \)-C**-algebra.


b) We have

\[ \langle x \otimes 1_K, (a, \xi, \eta) \rangle = \langle ((x \otimes 1_K)\xi|\eta), a \rangle = \left( \sum_{t \in T} \eta_t x \xi_t, a \right) = \sum_{t \in T} \langle \eta_t x \xi_t, a \rangle = \sum_{t \in T} \langle x \xi_t a_n^\tau, \eta \rangle. \]

Thus the family \( (\xi_t a_n^\tau)_{t \in T} \) is summable in \( E \) and

\[ \langle \varphi x, (a, \xi, \eta) \rangle = \left( x, \sum_{t \in T} \xi_t a_n^\tau \right). \]

If \( \varphi' : L_{\mathcal{E}}(H) \to E' \) denotes the transpose of \( \varphi \) then

\[ \varphi'(a, \xi, \eta) = \sum_{t \in T} \xi_t a_n^\tau \in \tilde{E}. \]

By continuity \( \varphi'(L_{\mathcal{E}}(H)) \subset E \) and \( \varphi \) is a unital W*-homomorphism.
c) Let $x \in E$ and $\xi, \eta \in \mathcal{L}_E(H)$. By [1] Proposition 5.6.3.17 d),
\[
\langle (x \overline{\otimes} 1_k)\xi|\eta \rangle = \sum_{t \in T} \eta^*_t (x \overline{\otimes} 1_k)\xi_t = \sum_{t \in T} \eta^*_t x\xi_t = \sum_{t \in T} x\eta^*_t \xi_t = x(\xi|\eta).
\]
Thus for $u \in \mathcal{L}_E(H)$,
\[
\langle u(x \overline{\otimes} 1_k)\xi|\eta \rangle = \langle (x \overline{\otimes} 1_k)\xi|u^*\eta \rangle = x(\xi|u^*\eta) = x(\xi|\eta),
\]
and so $x \overline{\otimes} 1_k \in \mathcal{L}_E(H)^c$.

**DEFINITION 1.3.3** We put for all $\xi, \eta \in H$ (resp. and $a \in \bar{E}_+$)
\[
P_{\xi,\eta} : \mathcal{L}_E(H) \rightarrow \mathbb{R}_+, \quad X \mapsto \|X\|, \\
(\text{ resp. } P_{\xi,\eta,a} : \mathcal{L}_E(H) \rightarrow \mathbb{R}_+, \quad X \mapsto \|X\|), \\
P_{\xi} : \mathcal{L}_E(H) \rightarrow \mathbb{R}_+, \quad X \mapsto \|X\| = \|X[H^\perp\xi]\|^1/2, \\
(\text{ resp. } P_{\xi,a} : \mathcal{L}_E(H) \rightarrow \mathbb{R}_+, \quad X \mapsto \|X[H^\perp\xi]\|^{1/2}), \\
q_{\xi} : \mathcal{L}_E(H) \rightarrow \mathbb{R}_+, \quad X \mapsto p_\xi(X^*), \\
(\text{ resp. } q_{\xi,a} : \mathcal{L}_E(H) \rightarrow \mathbb{R}_+, \quad X \mapsto p_{\xi,a}(X^*)).
\]
and denote, respectively, by $\Xi_1, \Xi_2, \Xi_3$ the topologies on $\mathcal{L}_E(H)$ generated by the set of seminorms
\[
\{ p_{\xi,\eta} \mid \xi, \eta \in H \}, \quad (\text{ resp. } \{ p_{\xi,\eta,a} \mid \xi, \eta \in H, a \in \bar{E}_+ \}), \\
\{ p_{\xi} \mid \xi \in H \}, \quad (\text{ resp. } \{ p_{\xi,a} \mid \xi \in H, a \in \bar{E}_+ \}), \\
\{ p_{\xi} \mid \xi \in H \} \cup \{ q_{\xi} \mid \xi \in H \}, \quad (\text{ resp. } \{ p_{\xi,a} \mid \xi \in H, a \in \bar{E}_+ \} \cup \{ q_{\xi,a} \mid \xi \in H, a \in \bar{E}_+ \}).
\]
Moreover $\|\cdot\|$ denotes the norm topology on $\mathcal{L}_E(H)$.

Of course $\Xi_3 \subset \Xi_2$. In the $\text{C}^*$-case, $\Xi_2$ is the topology of pointwise convergence. If $E$ is finite-dimensional then the $\text{C}^*$-case and the $W^*$-case coincide.

**PROPOSITION 1.3.4** Let $X \in \mathcal{L}_E(H)$ and $\xi, \eta \in H$ (resp. and $a \in \bar{E}$).
\[
\begin{align*}
a) & \quad p_{\xi,a}(X) = p_{\xi,a}(X^*) \quad (\text{ resp. } p_{\xi,a}(X) = p_{\xi,a}(X^*)). \\
b) & \quad p_{\xi,a}(X) \leq p_{\xi}(X) \|\eta\|. \\
c) & \quad \text{If } E \text{ is a } W^*\text{-algebra and } a = x|a| \text{ is the polar representation of } a \text{ then} \\
& \quad p_{\xi,a}(X) = |\langle X, (a,\overline{a}) \rangle| \leq p_{\xi,a}(X)(\|\eta\|, |a|)^{1/2}. \\
d) & \quad \text{If } Y, Z \in \mathcal{L}_E(H) \text{ then} \\
& \quad p_{\xi,a}(YZ) = p_{\xi,a}(Y^*Z(X)) \quad (\text{ resp. } p_{\xi,a}(YZ) = p_{\xi,a}(Y^*Z(X)), \\
& \quad p_{\xi}(YZ) \leq \|Y\|p_{\xi}(X) \quad (\text{ resp. } p_{\xi}(YZ) \leq \|Y\|p_{\xi}(X)).
\end{align*}
\]
We prove the relation by induction with respect to\[d)\] the first equation follows from
and the second from
\[p_{\xi,y}(X) = \|\langle X \xi|\eta\rangle,|a|\| = \|\langle X \xi|\eta\rangle,|a|\| = \|\langle X \xi|\eta\rangle,|a|\| = \|\langle X \xi|\eta\rangle,|a|\|.\]

By Schwarz’ inequality ([1] Proposition 2.3.3.9),
\[\|\langle X \xi|\eta\rangle,|a|\|^2 \leq \|\langle X \xi|X \xi\rangle\| |a| \|\langle \xi|\eta\rangle,|a|\|,\]
so
\[p_{\xi,y,|a|}(X) \leq p_{\xi,y}(X) |\langle \xi|\eta\rangle,|a||^{1/2}.\]

\[d)\] The first equation follows from
\[p_{\xi,y}(YXZ) = \|\langle Y X \xi|\eta\rangle\| = \|\langle X \xi|Y^* \eta\rangle\| = p_{\xi,y}(X)\]
( resp. \[p_{\xi,y,|a|}(Y X Z) = \|\langle Y X \xi|\eta\rangle,|a|\| = \|\langle X Z \xi|Y^* \eta\rangle,|a|\| = p_{\xi,y}(X)\])
and the second from
\[p_{\xi}(YXZ) = \|YXZ\| \leq \|Y\| \|XZ\| = \|Y\| p_{\xi}(X)\]
( resp. \[p_{\xi,|a|}(YXZ) = \|\langle YXZ\xi|YXZ\xi\rangle,|a|\|^{1/2} \leq \|Y\| \|\langle YXZ\xi|YXZ\xi\rangle,|a|\|^{1/2} = \|Y\| p_{\xi,|a|}(X)\}).

**LEMA 1.3.5** Let \( n \in \mathbb{N} \) and \( (x_i)_{i \in \mathbb{N}} \) a family in \( E \). Then
\[
\left( \sum_{i \in \mathbb{N}} x_i \right)^* \left( \sum_{i \in \mathbb{N}} x_i \right) \leq n \sum_{i \in \mathbb{N}} x_i^* x_i.
\]

We prove the relation by induction with respect to \( n \). By [1] Corollary 4.2.2.4 and by the hypothesis of the induction,
\[
\left( \sum_{i \in \mathbb{N}} x_i \right)^* \left( \sum_{i \in \mathbb{N}} x_i \right) = \left( x_n^* + \sum_{i \in \mathbb{N} \setminus \{n\}} x_i^* \right) \left( x_n + \sum_{i \in \mathbb{N} \setminus \{n\}} x_i \right) =
\]
\[
x_n x_n + \sum_{i \in \mathbb{N} \setminus \{n\}} (x_n^* x_i + x_i^* x_n) + \left( \sum_{i \in \mathbb{N} \setminus \{n\}} x_i \right)^* \left( \sum_{i \in \mathbb{N} \setminus \{n\}} x_i \right) \leq
\]
\[
\leq x_n^* x_n + \sum_{i \in \mathbb{N} \setminus \{n\}} (x_n^* x_i + x_i^* x_n) + (n-1) \sum_{i \in \mathbb{N} \setminus \{n\}} x_i^* x_i = n \sum_{i \in \mathbb{N}} x_i^* x_i.
\]
LEMMA 1.3.6 Let $n \in \mathbb{N}$, $x \in E_{n\theta}$, and for every $j \in \mathbb{N}_n$ put

$$\eta_j : (\delta_j)_{\mathbb{N}_n} \in \bigotimes_{j \in \mathbb{N}_n} \mathcal{E}.$$ 

Then

$$\|x\| \leq \sqrt{n} \sup_{j \in \mathbb{N}_n} \|x \eta_j\|.$$ 

For $\xi \in (\bigotimes_{j \in \mathbb{N}_n} \mathcal{E})^*$, by Lemma 1.3.5,

$$\langle x \xi | x \xi \rangle = \sum_{i \in \mathbb{N}_n} \langle x_i \xi_j | x_i \xi_j \rangle = \sum_{i \in \mathbb{N}_n} \left( \sum_{j \in \mathbb{N}_n} x_{i j} \xi_j \right)^* \left( \sum_{j \in \mathbb{N}_n} x_{i j} \xi_j \right) \leq$$

$$\leq n \sum_{i \in \mathbb{N}_n} \sum_{j \in \mathbb{N}_n} (x_{i j} \xi_j)^* x_{i j} \xi_j = n \sum_{i \in \mathbb{N}_n} \sum_{j \in \mathbb{N}_n} x_{i j}^* x_{i j} \xi_j =$$

$$= n \sum_{j \in \mathbb{N}_n} \xi_j^* \left( \sum_{i \in \mathbb{N}_n} x_{i j}^* x_{i j} \right) \xi_j.$$ 

For $i, j \in \mathbb{N}_n$,

$$\langle x \eta_j | x \eta_j \rangle = \sum_{k \in \mathbb{N}_n} x_{i k} \eta_{j k} = x_{i j},$$

$$\langle x \eta_j | x \eta_j \rangle = \sum_{i \in \mathbb{N}_n} (x_{i j})^* (x_{i j}) = \sum_{i \in \mathbb{N}_n} x_{i j}^* x_{i j},$$

so

$$\langle x \xi | x \xi \rangle \leq n \sum_{i \in \mathbb{N}_n} \xi_j^* \langle x \eta_j | x \eta_j \rangle \xi_j \leq n \sum_{i \in \mathbb{N}_n} \|x \eta_j\|^2 \xi_j^* \xi_j \leq$$

$$\leq n \sup_{j \in \mathbb{N}_n} \|x \eta_j\|^2 \sum_{j \in \mathbb{N}_n} \xi_j^* \xi_j \leq n \sup_{j \in \mathbb{N}_n} \|x \eta_j\|^2 1_\mathcal{E},$$

$$\|x\|^2 \leq n \sup_{j \in \mathbb{N}_n} \|x \eta_j\|^2.$$ 

\[\square\]

COROLLARY 1.3.7

a) The map

$$\mathcal{L}_E(H)_{\mathbb{Z}_1} \longrightarrow \mathcal{L}_E(H)_{\mathbb{Z}_1}, \quad X \longmapsto X^*$$

is continuous. In particular $\text{Re} \mathcal{L}_E(H)$ is a closed set of $\mathcal{L}_E(H)_{\mathbb{Z}_1}$.

b) $\mathbb{Z}_1 \subset \mathbb{Z}_2 \subset \mathbb{Z}_3 \subset \text{norm topology}$.

c) If $E$ is a $W^*$-algebra then the identity map

$$\mathcal{L}_E(H)_{\mathbb{N}} \longrightarrow \mathcal{L}_E(H)_{\mathbb{Z}_1}$$

is continuous so

$$\mathcal{L}_E(H)_{\mathbb{Z}_1} = \mathcal{L}_E(H)_{\mathbb{N}}$$

is compact.

d) For $Y, Z \in \mathcal{L}_E(H)$ and $k \in \{1, 2\}$, the map

$$\mathcal{L}_E(H)_{\mathbb{Z}_k} \longrightarrow \mathcal{L}_E(H)_{\mathbb{Z}_k}, \quad X \longmapsto YXZ$$

is continuous.

e) $\mathcal{L}_E(H)_{\mathbb{Z}_2}$ is complete in the $C^*$-case.

f) If $T$ is finite then $\mathbb{Z}_2$ is the norm topology in the $C^*$-case.

g) $K_E(H)$ is dense in $\mathcal{L}_E(H)_{\mathbb{Z}_1}$.
a) follows from Proposition 1.3.4 a).
b) \( T_1 \subset T_2 \) follows from Proposition 1.3.4 b,c). \( T_2 \subset T_3 \subset \) norm topology is trivial.
c) follows from Proposition 1.3.4 c) (and [1] Theorem 5.6.3.5 a)).
d) follows from Proposition 1.3.4 d).
e) Let \( \mathcal{F} \) be a Cauchy filter on \( \mathcal{L}_E(H)_{T_2} \). Put
\[
Y : H \rightarrow H, \quad \xi \mapsto \lim_{\mathcal{F}} (X\xi),
\]
\[
Z : H \rightarrow H, \quad \xi \mapsto \lim_{\mathcal{F}} (X^*\xi),
\]
where the limits are considered in the norm topology of \( H \). For \( \xi, \eta \in H \),
\[
(Y\xi|\eta) = \lim_{\mathcal{F}} (X\xi|\eta) = \lim_{\mathcal{F}} (\xi|X^*\eta) = (\xi|Z\eta),
\]
so \( Y, Z \in \mathcal{L}_E(H) \) and \( Z = Y^* \). Thus \( \mathcal{F} \) converges to \( Y \) in \( \mathcal{L}_E(H)_{T_2} \) and \( \mathcal{L}_E(H)_{T_3} \) is complete.
f) follows from b) and Lemma 1.3.6.
g) Let \( X \in \mathcal{L}_E(H) \) and \( \xi \in H \). For every \( S \in \Psi_f(T) \) put
\[
P_S := \sum_{s \in S} e_s \mathbb{C} = \sum_{s \in S} e_s \xi \mathbb{C} \in \mathcal{K}_E(H)
\]
and let \( \mathcal{F}_T \) be the upper section filter or \( \Psi_f(T) \). Then \( P_S X \in \mathcal{K}_E(H) \) for every \( S \in \Psi_f(T) \) and
\[
\lim_{S, \mathcal{F}_T} P_S X = X
\]
in \( H \) (resp. in \( H_H \)) ([1] Proposition 5.6.4.1 e) (resp. [1] Proposition 5.6.4.6 c)). Thus
\[
\lim_{S, \mathcal{F}_T} P_S X = X
\]
with respect to the topology \( T_2 \). Since the same holds for \( X^* \), it follows that \( X \) belongs to the closure of \( \mathcal{K}_E(H) \) in \( \mathcal{L}_E(H)_{T_2} \).

**Remark.** The inclusions in b) can be strict as it is known from the case \( E = \mathbb{K} \).

**Lemma 1.3.8** Let \( G \) be a W*-algebra and \( F \) a C*-subalgebra of \( G \). Then the following are equivalent.

a) \( F \) generates \( G \) as a W*-algebra.
b) \( F^* \) is dense in \( G^* \).
c) \( F \) is dense in \( G \).

\( a \Rightarrow b \) follows from [1] Corollary 6.3.8.7.
\( b \Rightarrow c \) is trivial.
\( c \Rightarrow a \) follows from [1] Corollary 4.4.4.12 a).

**Proposition 1.3.9** Let \( G \) be a W*-algebra, \( F \) a C*-subalgebra of \( G \) generating it as W*-algebra, \( I \) a set, and
\[
L := \bigoplus_{i \in I} \tilde{F}, \quad M := \bigoplus_{i \in I} \tilde{G}.
\]

a) \( M \) is the extension of \( L \) to a selfdual Hilbert right \( G \)-module ([2] Proposition 1.3 f)) and \( L^* \) is dense in \( M^*_M \).
b) If we denote for every \( X \in \mathcal{L}_F(L) \) by \( \bar{X} \in \mathcal{L}_G(M) \) its unique extension ([3] Proposition 1.4 a)) then the map
\[
\mathcal{L}_F(L) \rightarrow \mathcal{L}_G(M), \quad X \mapsto \bar{X}
\]
is an injective C*-homomorphism and its image is dense in \( \mathcal{L}_G(M)_{\bar{M}} \).
c) The map
\[
\mathcal{L}_F(L)^*_{T_2} \rightarrow \mathcal{L}_G(M)^*_{T_2}, \quad X \mapsto \bar{X}
\]
is continuous.
a) By Lemma 1.3.8 $a \Rightarrow b$, $P^a$ is dense in $G^a$, so $\tilde{F}^a$ is dense in $\tilde{G}^a$ and $\tilde{G}$ is the extension of $\tilde{F}$ to a selfdual Hilbert right $G$-module. ([3] Corollary 1.5 $a_2 \Rightarrow a_1$). By [3] Proposition 1.8, $M$ is the extension of $L$ to a selfdual Hilbert right $G$-module. By [3] Corollary 1.5 $a_1 \Rightarrow a_2$, $L^a$ is dense in $M^a$.

b) By a) and [3] Proposition 1.4 e), the map

$$\mathcal{L}_F(L) \longrightarrow \mathcal{L}_C(M), \quad X \longmapsto \tilde{X}$$

is an injective C*-homomorphism. By [3] Proposition 1.9 b), its image is dense in $\mathcal{L}_C(M)_{\tilde{\mathcal{M}}}$.

c) Denote by $N$ the vector subspace of $\tilde{M}$ generated by

$$\{(a, \tilde{\xi}, \eta) \mid (a, \xi, \eta) \in \tilde{G} \times L \times L\}.$$  

By a) and [3] Proposition 1.9 a), $N$ is dense in $\tilde{M}$ so by Corollary 1.3.7 c),

$$\mathcal{L}_C(M)_{\tilde{\mathcal{M}}} = \mathcal{L}_C(M)_{\mathcal{M}}.$$  

For $(a, \xi, \eta) \in \tilde{G} \times L \times L$ and $X \in \mathcal{L}_F(L)$, by Proposition 1.3.4 c),

$$p_{\xi, \eta, a}(X) = |\langle (\tilde{X}\xi)\eta, a \rangle| = |\langle (X\xi)\eta, a \rangle| \leq p_{\xi, \eta}(X)\langle (\eta, a)\rangle^2,$$

where $a = x[a]$ is the polar representation of $a$, so the map

$$\mathcal{L}_F(L)_{\mathcal{M}} \longrightarrow \mathcal{L}_C(M)_{\mathcal{M}}, \quad X \longmapsto \tilde{X}$$

is continuous.

**Lemma 1.3.10** Let $n \in \mathbb{N}$, $\xi \in \bigotimes_{i \in \mathbb{N}_n} E$, and

$$x = [\xi, \delta_i 1_{E}]_{j \in \mathbb{N}_n} \in E_{a, n}.$$  

Then $\|x\| = \|\xi\|$.  

For $\eta \in \bigotimes_{i \in \mathbb{N}_n} E$ and $i \in \mathbb{N}_n$,

$$\langle x\eta \rangle_i = \sum_{j \in \mathbb{N}_n} x_i\eta_j = \sum_{j \in \mathbb{N}_n} \xi_i\delta_j 1_{E}\eta_j = \xi_i \eta_1,$$

$$\langle x\eta | x\eta \rangle = \sum_{i \in \mathbb{N}_n} \langle (x\eta)_i | (x\eta)_i \rangle = \sum_{i \in \mathbb{N}_n} \xi_i^* \xi_i \eta_1 = \eta_1^* \left( \sum_{i \in \mathbb{N}_n} \xi_i^* \xi_i \right) = \eta_1^* (\xi^* \xi) \eta_1 \leq \|\xi\|^2 \|\eta_1\|^2,$$

$$\|x\eta\|^2 \leq \|\xi\|^2 \|\eta_1\|^2 \leq \|\xi\|^2 \|\eta_1\|^2, \quad \|x\| \leq \|\xi\|.$$  

On the other hand if we put $\zeta = (\delta_i 1_{E})_{i \in \mathbb{N}_n}$ then for $i \in \mathbb{N}_n$,

$$\langle x\zeta \rangle_i = \sum_{j \in \mathbb{N}_n} x_i\zeta_j = \sum_{j \in \mathbb{N}_n} \xi_i\delta_j 1_{E} = \xi_i,$$

$$\langle x\zeta | x\zeta \rangle = \sum_{i \in \mathbb{N}_n} \langle (x\zeta)_i | (x\zeta)_i \rangle = \sum_{i \in \mathbb{N}_n} \xi_i^* \xi_i = \|\xi\|^2,$$

$$\|x\zeta\|^2 = \|\xi\|^2, \quad \|x\| = \|\xi\|.$$  

**Lemma 1.3.11** Let $F, G$ be unital C*-algebras, $\varphi : F \rightarrow G$ a surjective C*-homomorphism, $I$ a set,

$$L := \bigcap_{i \in I} F \cong \bigotimes_{i \in I} F^i (I), \quad M := \bigcap_{i \in I} G \cong \bigotimes_{i \in I} G^i (I),$$

and for every $\xi \in L$ put $\tilde{\xi} := (\varphi^i \xi)_{i \in I}$. 

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a) If \( \xi, \eta \in L \) and \( x \in F \) then

\[
\xi \in M, \quad \|\xi\| \leq \|\xi\|. \quad (\xi\eta) = (\xi)\varphi x, \quad (\xi|\eta) = \varphi(\xi|\eta).
\]

b) For every \( \eta \in M \) there is a \( \xi \in L \) with \( \xi = \eta \), \( \|\xi\| \leq \|\eta\| \).

c) In the W*-case the map

\[
L_L \rightarrow M_M, \quad \xi \mapsto \tilde{\xi}
\]

is continuous.

a) For \( j \in \Psi_f(I) \),

\[
\sum_{i \in I} \langle \varphi \xi_i \varphi \eta_i \rangle = \sum_{i \in I} \langle \varphi \eta_i \rangle (\varphi \xi_i) = \varphi \sum_{i \in I} \eta_i^* \xi_i.
\]

It follows \( \tilde{\xi} \in M \), \( \|\tilde{\xi}\| \leq \|\xi\| \), \( \tilde{\xi}\eta = \varphi(\xi \eta) \). Moreover for \( i \in I \),

\[
(\tilde{\xi} x)_i = \varphi(x)_i \varphi(\xi_i) = (\varphi(\xi_i)) = \tilde{\xi}(x), \quad \tilde{\xi} x = \tilde{\xi}(x).
\]

b) Case 1 \( \{i \in I | \eta_i \neq 0\} \) is finite
For simplicity we assume \( \{i \in I | \eta_i \neq 0\} = N \) for some \( n \in \mathbb{N} \). We put

\[
\theta : F_{n,n} \rightarrow G_{n,n}, \quad [x_{ij}]_{i \in N \in n} \mapsto [\varphi x_{ij}]_{i \in N \in n}.
\]

\( \theta \) is obviously a surjective C*-homomorphism. So if we put

\[
y := [\eta_i \delta_{i1}]_{i \in N \in n} \in G_{n,n},
\]

then there is an \( x \in F_{n,n} \) with \( \theta x = y \), \( \|x\| \leq \|y\| \) (15) Theorem 10.1.7). If we put

\[
\xi : I \rightarrow \tilde{\xi}, \quad i \mapsto \left\{ \begin{array}{ll} x_i & \text{if } i \in \mathbb{N} \in n, \\ 0 & \text{if } i \in I \setminus \mathbb{N} \in n, \end{array} \right.
\]

and \( z := [x_i \delta_{i1}]_{i \in N \in n} \in F_{n,n} \) then

\[
\theta z = [\varphi(x_i \delta_{i1})]_{i \in N \in n} = [\varphi y_i \delta_{i1}]_{i \in N \in n} = y
\]

and by [1] Theorem 5.6.6.1 a), \( \|z\| \leq \|x\| \). We get for \( i \in \mathbb{N} \in n \),

\[
\tilde{\xi}_i = \varphi \xi_i = \varphi x_n = y_n = \eta_i.
\]

By a) and Lemma 1.3.10,

\[
\|\xi\| = \|z\| \leq \|x\| = \|y\| = \|\eta\| = \|\tilde{\xi}\| \leq \|\xi\|, \quad \|\xi\| = \|\eta\|.
\]

Case 2 \( \eta \) arbitrary in the W*-case
We may assume \( \|\eta\| = 1 \). We put for every \( j \in \Psi_f(I) \),

\[
\eta_j : I \rightarrow G, \quad i \mapsto \left\{ \begin{array}{ll} \eta_j & \text{if } i \in j, \\ 0 & \text{if } i \in I \setminus j. \end{array} \right.
\]

By Case 1, for every \( j \in \Psi_f(I) \) there is a \( \xi_j \in L \) with \( \tilde{\xi}_j = \eta_j \) and \( \|\xi_j\| = \|\eta_j\| \leq 1 \). Let \( \mathcal{F} \) be an ultrafilter on \( \Psi_f(I) \) finer than the upper section filter of \( \Psi_f(I) \). By [1] Proposition 5.6.3.3 a \( \Rightarrow \) b,

\[
\xi := \lim_{j \in \mathcal{F}} \xi_j
\]

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exists in $L^2_L$. For $i \in I$,
$$\xi_i = \phi \xi_i = \phi \lim_{j \to i} (\xi_j)_i = \lim_{j \to i} \phi (\xi_j)_i = \eta_i,$$
so $\check{\xi} = \eta$. By a), $1 = ||\eta|| = ||\check{\xi}|| \leq ||\xi|| \leq 1$, so $||\xi|| = ||\eta||$.

Case 3 $\eta$ arbitrary in the C*-case
We put for every $J \in \Psi_I(I)$ and every $\zeta \in M$,
$$\zeta_J : I \rightarrow G, \quad i \rightarrow \begin{cases} \xi_i & \text{if } i \in J \\ 0 & \text{if } i \in I \setminus J \end{cases}.$$
Moreover we denote by $\mathcal{F}_J$, the upper section filter of $\Psi_I(I)$, set
$$M_0 := \{ \xi \in M | \{ i \in I | \xi_i \neq 0 \} \text{ is finite} \},$$
and denote by $M$ the vector subspace of $K_G(M)$ generated by the set
$$\{ \xi_1, \xi_2 | \xi_1, \xi_2 \in M_0 \}.$$
Let $G$ be the vector subspace of $K_G(L)$ generated by the set
$$\{ \alpha | \beta \} | \alpha, \beta \in L \}.$$
$G$ is an involutive subalgebra of $K_G(L)$. Let $(\alpha_q)_{q \in Q} \in \phi \ (\beta_q)_{q \in Q}$ be finite families in $L$ such that
$$\sum_{q \in Q} \alpha_q (\cdot | \beta_q) = 0.$$
and let \( \xi \in M_0^\mathbb{R} \). By Case 1, there is an \( \alpha \in L^\mathbb{R} \) with \( \alpha^* = \xi \). By a),

\[
\langle \psi \mu \rangle \xi = \sum_{q \in \mathbb{Q}} \tilde{a}_q (\tilde{a} \mid \beta_q) = \sum_{q \in \mathbb{Q}} \tilde{a}_q \psi(\alpha \mid \beta_q) = \sum_{q \in \mathbb{Q}} \tilde{a}_q (\alpha \mid \beta_q) = \tilde{u} \alpha,
\]

\[
\| \psi \mu \xi \| = \| \tilde{u} \alpha \| \leq \| u \alpha \| \leq \| u \|.
\]

Since \( M_0 \) is dense in \( M \) ([1] Proposition 5.6.4.1 e)), it follows

\[
\| \psi u \| \leq \| u \|, \quad \| \psi \| \leq 1.
\]

Step 2 \( M \) is dense in \( K_\mathbb{C}(M) \)

Let \( \alpha, \beta \in M \). By [1] Proposition 5.6.4.1 e),

\[
\alpha = \lim_{j \to 0} \alpha_j, \quad \beta = \lim_{j \to 0} \beta_j
\]

so by [1] Proposition 5.6.5.2 a),

\[
\alpha(\beta) = \lim_{j \to 0} \alpha_j(\beta_j),
\]

which proves the assertion.

Step 3 \( \psi \) is a surjective \( \mathbb{C}^* \)-homomorphism

By Step 1, \( \psi \) is a \( \mathbb{C}^* \)-homomorphism. Since its image contains \( M \) (by Case 1) it is surjective by Step 2.

Step 4 The assertion

Let \( j \in I \). By Step 3 and [5] Theorem 10.1.7 (and [1] Proposition 5.6.5.2 a)), there is a \( u \in K_\mathbb{F}(L) \) with

\[
\psi u = \eta(1_\mathbb{C} \otimes e_j), \quad \| u \| = \| \eta(1_\mathbb{C} \otimes e_j) \| = \| \eta \|.
\]

From

\[
\psi(u((1_\mathbb{F} \otimes e_j)(\cdot \mid 1_\mathbb{F} \otimes e_j))) = (\eta(1_\mathbb{C} \otimes e_j))(1_\mathbb{C} \otimes e_j)(\cdot \mid 1_\mathbb{C} \otimes e_j)) = \eta(1_\mathbb{C} \otimes e_j),
\]

\[
\| \eta \| = \| \eta(1_\mathbb{C} \otimes e_j) \| \leq \| u((1_\mathbb{F} \otimes e_j)(\cdot \mid 1_\mathbb{F} \otimes e_j)) \| \leq \| u \| \| (1_\mathbb{C} \otimes e_j)(\cdot \mid 1_\mathbb{F} \otimes e_j) \| = \| u \| = \| \eta \|.
\]

we see that we may assume

\[
u = u((1_\mathbb{C} \otimes e_j)(\cdot \mid 1_\mathbb{C} \otimes e_j)).
\]

Then

\[
u = (u(1_\mathbb{F} \otimes e_j))(\cdot \mid 1_\mathbb{F} \otimes e_j).
\]

If we put \( \xi := u(1_\mathbb{F} \otimes e_j) \in L \) then \( u = \xi (\cdot \mid 1_\mathbb{F} \otimes e_j) \| u \| = \| u \| = \| \xi \|.
\]

\[
\eta(1_\mathbb{C} \otimes e_j) = \psi u = \tilde{\xi}(\cdot \mid 1_\mathbb{C} \otimes e_j),
\]

\[
\eta = \eta(1_\mathbb{C} \otimes e_j \mid 1_\mathbb{C} \otimes e_j) = \tilde{\xi}(1_\mathbb{C} \otimes e_j \mid 1_\mathbb{C} \otimes e_j) = \tilde{\xi}.
\]

c) Let \( (a, \eta_0) \in \hat{G} \times M \). By b), there is an \( \tilde{\xi}_0 \in L \) with \( \tilde{\xi}_0 = \eta_0 \). By a), for \( \xi \in L \),

\[
\langle \xi, (a, \eta_0) \rangle = \langle (\xi, \eta_0), a \rangle = \langle (\xi, \tilde{\xi}_0), a \rangle = \langle \phi(\xi, \eta_0), a \rangle = \langle (\xi, \tilde{\xi}_0), \phi a \rangle = \langle \xi, (\phi a, \tilde{\xi}_0) \rangle.
\]
We put 

$$\theta : L \rightarrow M, \quad \xi \mapsto \tilde{\xi}$$

and denote by $\theta' : M' \rightarrow L'$ its transpose. By the above, $\theta'(a, \eta_0) \in \tilde{L}$. Since $\theta'$ is continuous, $\theta'(\tilde{M}) \subseteq \tilde{L}$ and this proves the assertion.

**PROPOSITION 1.3.12** We use the notation of Lemma 1.3.11.

a) If $X \in \mathcal{L}_F(L)$ and $\xi \in L$ with $\tilde{\xi} = 0$ then $X \tilde{\xi} = 0$; we define

$$\tilde{X} : M \rightarrow M, \quad \eta \mapsto \tilde{\eta}.$$ 

where $\xi \in L$ with $\tilde{\xi} = \eta$ (Lemma 1.3.11 b)).

b) For every $X \in \mathcal{L}_F(L)$, $\tilde{X}$ belongs to $\mathcal{L}_G(M)$ and the map

$$\mathcal{L}_F(L) \rightarrow \mathcal{L}_G(M), \quad X \mapsto \tilde{X}$$

is a surjective $C^\ast$-homomorphism continuous with respect to the topologies $\Sigma_k$ with $k \in \{1,2,3\}$.

c) For $\xi, \eta \in L$,

$$\hat{\eta}(\xi) = \eta(\xi)$$

and

$$K_G(M) = \{ \tilde{X} \mid X \in \mathcal{K}_F(L) \}.$$ 

a) For $i \in I$, $\varphi \xi_i = \tilde{\xi}_i = 0$ so by Lemma 1.3.11 a),

$$X(e_i \xi_i) = (Xe_i) \xi_i = (Xe_i) \varphi \xi_i = 0.$$

By [1] Proposition 5.6.4.1 e) (resp. [1] Proposition 5.6.4.6 c) and [1] Proposition 5.6.3.4 c)),

$$X \xi = X \left( \sum_{i \in I} e_i \xi_i \right) = \sum_{i \in I} X(e_i \xi_i)$$

$$\left( \text{resp } X \xi = X \left( \sum_{i \in I} e_i \xi_i \right) = \sum_{i \in I} X(e_i \xi_i) \right)$$

so by Lemma 1.3.11 a) (resp. c)),

$$\tilde{X} \xi = \sum_{i \in I} X(e_i \xi_i) = \sum_{i \in I} X(e_i \xi_i) = 0$$

$$\left( \text{resp } \tilde{X} \xi = \sum_{i \in I} X(e_i \xi_i) = \sum_{i \in I} X(e_i \xi_i) = 0 \right).$$

b) For $X, Y \in \mathcal{L}_F(L)$ and $\xi, \eta \in L$, by Lemma 1.3.11 a),

$$<\tilde{X} \xi, \tilde{\eta}> = \langle \tilde{X} \xi, \tilde{\eta} \rangle = \varphi (X \xi | \eta) =$$

$$= \varphi (\xi | X^\ast \eta) = <\tilde{\xi}, \tilde{X}^\ast \eta> = <\tilde{\xi}, \tilde{X}^\ast \eta>.$$ 

By Lemma 1.3.11 b), $\tilde{X} \in \mathcal{L}_G(M)$, $(\tilde{X})^\ast = \tilde{X}^\ast$, and $\tilde{XY} = \tilde{X} \tilde{Y}$, i.e. the map is a $C^\ast$-homomorphism.
For \( X \in \mathcal{L}_t(L) \) and \( \xi, \eta \in L \) (resp. and \( a \in \mathcal{M}_t \)), by Lemma 1.3.11 a),
\[
 p_{\xi,\eta}(X) = \|((\xi \otimes \eta) \xi)\| = \|((\xi \otimes \eta) \eta)\| = \|\phi(X \xi | \eta)\| \leq p_{\xi,\eta}(X)
\]
( resp. \( p_{\xi,\eta}(X) = \|((\xi \otimes \eta) \xi, a)\| = \|\phi(X \xi | \eta)\| a\) =
\[
 = \|((X \xi | \eta), \phi a)\| = p_{\xi,\eta,\phi}(X),
\]
so by Lemma 1.3.11 b), the map is continuous with respect to the topology \( \Xi_1 \). The proof for the other topologies is similar.

c) For \( \zeta \in L \), by Lemma 1.3.11 a),
\[
 \overline{\eta(\cdot | \xi)} \zeta = (\overline{\eta(\cdot | \xi)}) \zeta = \overline{\eta(\xi | \xi)} = \\
 = \overline{\eta \phi(\zeta | \xi)} = \overline{\eta \tilde{\xi} | \tilde{\xi}} = (\overline{\eta(\cdot | \tilde{\xi} \tilde{\xi})}
\]
so by Lemma 1.3.11 b),
\[
 \overline{\eta(\cdot | \tilde{\xi})} = \overline{\eta(\cdot | \tilde{\xi})}
\]
The last assertion follows now from b).

**SCHUR PRODUCTS**

Throughout this chapter we fix \( f \in \mathcal{F}(T, E) \)

### 2.1 The representations

We present here the projective representation of the groups and its main properties.

**DEFINITION 2.1.1** We put for every \( t \in T \) and \( \xi \in H \),
\[
 u_t : \bar{E} \longrightarrow H, \quad \xi \mapsto \xi \otimes e_t, \\
 V_t : T \longrightarrow \bar{E}, \quad s \mapsto f(t, t^{-1} s) \xi(t^{-1} s) .
\]

If we want to emphasize the role of \( f \) then we put \( V_t^f \) instead of \( V_t \). For \( x \in E \),
\[
 (x \otimes 1_K) V_t : T \longrightarrow \bar{E}, \quad s \mapsto f(t, t^{-1} s) x \xi(t^{-1} s) .
\]

**PROPOSITION 2.1.2** Let \( s, t \in T, x \in E, \xi \in \bar{E}, \) and \( \zeta \in H \).

a) \( V_t \xi \in H \).

b) \( V_t V_t = (f(s, t) \otimes 1_K) V_{st} \).

c) \( V_t (\zeta \otimes e_t) = (f(t, s) \zeta) \otimes e_t \).

d) \( V_t (x \otimes 1_K) = (x \otimes 1_K) V_t \).

e) \( V_t \in L E(H), \quad V_t^f = (f(t) \otimes 1_K) V_{t^{-1}} \).

f) \( (x \otimes 1_K) V_t (\zeta \otimes e_t) = (f(t, s) x \zeta) \otimes e_{st} \).

g) If \( T \) is infinite and \( \mathfrak{F} \) denotes the filter on \( T \) of cofinite subsets, i.e.
\[
 \mathfrak{F} = \{ S \mid S \in \mathcal{B}(T), \quad T \setminus S \in \mathcal{B}(T) \},
\]

then

\[
 \mathfrak{F} = \{ S \mid S \in \mathcal{B}(T), \quad T \setminus S \in \mathcal{B}(T) \}.
\]
\[ \lim_{\varepsilon \to 0} V_\varepsilon = 0 \]

in \( \mathcal{L}_\varepsilon(H) \) of \( \varepsilon \).

a) For \( \varepsilon = B \in \Psi_f(T) \),
\[ \sum_{r \in B} \langle (V_\varepsilon \xi), (V_\varepsilon \zeta) \rangle = \sum_{r \in B} (f(t, t^{-1} r) \xi_{t^{-1} r}, f(t, t^{-1} r) \zeta_{t^{-1} r}) = \sum_{r \in B} \langle \xi_{t^{-1} r}, \zeta_{t^{-1} r} \rangle = \sum_{r \in B} (\xi_{t^{-1} r}, \xi_{t^{-1} r}) = \sum_{r \in B} (\zeta_{t^{-1} r}, \zeta_{t^{-1} r}) \]
so \( V_\varepsilon \xi \in H \).

b) For \( r \in T \),
\[ (V_\varepsilon V_\xi) r = f(s, s^{-1} r) (V_\xi)_{t^{-1} r} = f(s, t^{-1} r) f(t, t^{-1} s^{-1} r) \xi_{t^{-1} r} = f(s, t) f(st, t^{-1} s^{-1} r) \xi_{t^{-1} r} = f(s, t) (V_\varepsilon \xi) r = ((f(t, s) \otimes 1_k) V_\varepsilon \xi) r \]
so
\[ V_\varepsilon V_\xi = (f(s, t) \otimes 1_k) V_\varepsilon . \]

c) For \( r \in T \),
\[ (V_\varepsilon (\zeta \otimes e_0)) r = f(t, t^{-1} r) (\zeta \otimes e_0)_{t^{-1} r} = \delta_{t, t}(f(t, t^{-1} r) \xi) = \delta_{t, t} f(t, t^{-1} r) \xi = (f(t, t) \otimes e_0) r \]
so
\[ V_\varepsilon (\zeta \otimes e_0) = (f(t, t) \xi) \otimes e_0 . \]

d) We have
\[ (V_\varepsilon (x \otimes 1_k) \xi) r = f(t, t^{-1} s) (x \otimes 1_k) \xi_{t^{-1} s} = f(t, t^{-1} s) x \xi_{t^{-1} s} = ((x \otimes 1_k) V_\varepsilon \xi) r \]
so
\[ V_\varepsilon (x \otimes 1_k) = (x \otimes 1_k) V_\varepsilon . \]

e) For \( \eta \in H \), by Proposition 1.1.2 a),b),
\[ (V_\varepsilon \xi | \eta) = \sum_{t \in T} \langle (V_\varepsilon \xi), (V_\varepsilon \eta) \rangle = \sum_{t \in T} (f(t, t^{-1} s) \xi_{t^{-1} s}, \eta) = \sum_{t \in T} \langle \xi, f(t, t^{-1} s) \eta \rangle = \sum_{t \in T} \langle \xi, f(t, t^{-1} s) \eta \rangle = \sum_{t \in T} \langle \xi, ((f(t) \otimes 1_k) V_\varepsilon \xi) r \rangle = \langle \xi, ((f(t) \otimes 1_k) V_\varepsilon \xi) r \rangle \]
so \( V_\varepsilon \in \mathcal{L}_\varepsilon(H) \) with \( V_\varepsilon^* = (f(t) \otimes 1_k) V_{t^{-1}} \). By b) and d),
\[ V_\varepsilon^* V_\varepsilon = (f(t) \otimes 1_k) V_{t^{-1}} V_\varepsilon = (f(t) \otimes 1_k) (f(t^{-1}, t) \otimes 1_k) V_{t^{-1}} = id_H, \]

f) follows from c).

g) Let us consider first the C*-case. Let \( \varepsilon, \eta \in H, t \in T, \) and \( \varepsilon > 0 \). There is an \( S \in \Psi_f(T) \) such that \( \| \eta \eta_{t^{-1} s} \| < \varepsilon \). By e),
\[ |(V_\varepsilon \xi | \eta \eta_{t^{-1} s})| \leq \| V_\varepsilon \xi \| \| \eta \eta_{t^{-1} s} \| \leq \varepsilon \| \xi \| \]
so
\[ p_{\varepsilon, \eta}(V_\varepsilon) = |(V_\varepsilon \xi | \eta) - |(V_\varepsilon \xi | \eta \eta_{t^{-1} s})| + |(V_\varepsilon \xi | \eta \eta_{t^{-1} s})| < |(V_\varepsilon \xi | \eta \eta_{t^{-1} s})| + \varepsilon . \]

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From
\[ (V_t \xi | \eta e_s) = \sum_{s \in S} \eta^*_s f(t, t^{-1}s) \xi_{t^{-1}s} \]

it follows
\[ \lim_{t \to \infty} (V_t \xi | \eta e_s) = 0, \quad \lim_{t \to \infty} p_\xi(V_t) = 0. \]

The W*-case can be proved similarly.

\textit{Remark.} By e), \( \mathcal{T}_1 \) cannot be replaced by \( \mathcal{T}_2 \) in g).

**PROPOSITION 2.1.3** Let \( s, t \in T \).

a) \( u_s \in \mathcal{L}_E(\mathcal{E}, H) \), \( u_s^* = (\cdot | 1_E \otimes e_t) \).

b) \( u_s^* u_t = \delta_{s,t} 1_E \).

c) \( u_s^* u_t^* = 1_E \mathcal{S}((\cdot | e_t) e_s) \).

d) \( \sum_{t \in T} u_s^* u_t^* = id_H \).

a) For \( \zeta \in \mathcal{E} \) and \( \xi \in H \),
\[ \langle u_t \zeta, \xi \rangle = (\zeta \otimes e_t | \xi) = \sum_{s \in T} \xi_s (\zeta \otimes e_s) = \xi_t \zeta = (\zeta | \xi) \]
so
\[ u_t \in \mathcal{L}_E(\mathcal{E}, H), \quad u_t^* \xi = (\xi | 1_E \otimes e_t). \]

b) For \( \zeta \in \mathcal{E} \), by a),
\[ u_t^* u_s \zeta = u_s^* (\zeta \otimes e_t) = (\zeta \otimes e_t | 1_E \otimes e_s) = \delta_{s,t} \zeta \]
so \( u_s^* u_t = \delta_{s,t} 1_E \).

c) For \( \zeta \in \mathcal{E} \) and \( r \in T \), by a),
\[ u_s^* u_r^* (\zeta \otimes e_r) = u_r \delta_{s,r} \zeta = (\zeta \otimes e_r) = \delta_{s,r} \zeta = (\zeta \otimes e_r) \]
so (by a) and [1] Proposition 5.6.4.1 e) (resp. and [1] Proposition 5.6.4.6 c), [1] Proposition 5.6.3.4 c)) \( u_s^* u_t^* = 1_E \mathcal{S}((\cdot | e_t) e_s) \).

d) For \( \xi \in H \) (resp. and \( a \in \mathcal{E}_+ \)) and \( S \in \mathcal{P}_f(T) \), by c),
\[ p_\xi \left( \sum_{t \in S} u_s^* u_t^* - id_H \right) = \left\| \sum_{t \in T \setminus S} \langle \xi | \xi \rangle \right\|^{1/2} \]
\[ \text{resp.} \quad p_{\xi, a} \left( \sum_{t \in S} u_s^* u_t^* - id_H \right) = \left\| \sum_{t \in S} \langle \xi | \xi \rangle \right\|^{1/2} \]
\[ = \left( \sum_{t \in S} \langle \xi | \xi \rangle, a \right)^{1/2} \]

and the assertion follows.

**PROPOSITION 2.1.4** Let \( s, t \in T \) and \( x \in E \).

a) \( V_s u_t = u_{st} f(s, t) \).

b) \( u_s^* V_t = f(t, t^{-1}s) u_t^* u_s^* \).
c) \((x \otimes 1_K)u_t = u_x x_t\).

d) \(xu^*_t = u^*_t (x \otimes 1_K)\).

a) For \(\xi \in \hat{E}\), by Proposition 2.1.2 c),

\[
V_s u_t \xi = V_s (\xi \otimes e_t) = (f(s, t) \xi) \otimes e_t = u_t f(s, t) \xi
\]

so \(V_s u_t = u_t f(s, t)\).

b) For \(\xi \in \hat{E}\) and \(r \in T\), by Proposition 2.1.2 c) and Proposition 2.1.3 a),

\[
u^*_t V_t (\xi \otimes e_t) = u^*_t ((f(t, r) \xi) \otimes e_r) = \delta_{r, rf(t, r)} = \delta_{r, tf^{-1}(s)} = f(t, t^{-1}s) u^*_{t^{-1} s}(\xi \otimes e_t)
\]

so \(u^*_t V_t = f(t, t^{-1}s) u^*_{t^{-1} s}\).

c) For \(\xi \in \hat{E}\),

\[(x \otimes 1_K)u_t \xi = (x \otimes 1_K)(\xi \otimes e_t) = (x \xi) \otimes e_t = u_x \xi
\]

so \((x \otimes 1_K)u_t = u_x x_t\).

d) follows from c).

**DEFINITION 2.1.5** We put for all \(s, t \in T\) (Proposition 2.1.3 a))

\[
\varphi_{s,t} : L_\hat{E}(H) \to L_\hat{E}(\hat{E}) \cong \hat{E}, \quad X \mapsto u^*_t X u_s
\]

and set \(X_t := \varphi_{s, 1} X\) for every \(X \in L_\hat{E}(H)\).

**PROPOSITION 2.1.6** Let \(s, t \in T\).

a) \(\varphi_{s,t}\) is linear with \(\|\varphi_{s,t}\| = 1\).

b) For \(X \in L_\hat{E}(H)\) and \(x, y \in \hat{E}\),

\[
\langle (\varphi_{s,t} X) x | y \rangle = \langle X (x \otimes e_t) | y \otimes e_s \rangle.
\]

c) The map

\[
\varphi_{s,t} : L_\hat{E}(H) \to \hat{E} \quad (\text{resp.} \ E_{\hat{E}})
\]

is continuous.

d) \(\varphi_{s,t}\) is involutive and completely positive.

e) For \(r \in T\) and \(x \in \hat{E}\),

\[
\varphi_{s,t}((x \otimes 1_K) V_r) = \delta_{r, rf(t, t^{-1}s)} x.
\]

f) If \((x_t)_r \in T \in E(T)\) and

\[
X := \sum_{r \in T} (x_r \otimes 1_K) V_r
\]

then

\[
\varphi_{s,t} X = f(st^{-1}, t)x_{st^{-1}}, \quad X_t = x_t.
\]
g) For $X \in \mathcal{L}_E(H)$ and $x, y \in E$,

$$\varphi_{z,t}((x \otimes 1_k)X(y \otimes 1_k)) = x(\varphi_{z,t}X)y,$$

$$((x \otimes 1_k)X(y \otimes 1_k))_t = xX_t y.$$ 

a) follows from Proposition 2.1.3 a),b).

b) We have

$$\langle (\varphi_{z,t}X)x|y \rangle = \langle u_t^* Xu_t x|y \rangle = \langle Xu_t u_t y \rangle = \langle X(x \otimes e_1)|y \otimes e_2 \rangle.$$ 

c) The $C^*$-case

By b), for $X \in \mathcal{L}_E(H)$,

$$\|\varphi_{z,t}X\| = \|\langle (\varphi_{z,t}X)1_E|1_E \rangle \| =$$

$$= \|X(1_E \otimes e_1)|1_E \otimes e_2 \rangle\| = p_{1E \otimes e_1, 1E \otimes e_2}(X).$$

The $W^*$-case

Let $a \in \hat{E}$ and let $a = x|a|$ be its polar representation. By b), for $X \in \mathcal{L}_E(H)$,

$$\|\varphi_{z,t}X\| = \|\langle (\varphi_{z,t}X)1_E|1_E \rangle \| =$$

$$= \|\langle (X(x \otimes e_1)|1_E \otimes e_2 \rangle, |a\rangle \| = p_{x \otimes e_1, 1E \otimes |a\rangle}(X).$$ 

d) For $X \in \mathcal{L}_E(H)$,

$$(\varphi_{z,t}X)^* = (u_t^* Xu_t)^* = u_t^* X^* u_t = \varphi_{z,t}(X^*)$$ 

so $\varphi_{z,t}$ is involutive. For $n \in \mathbb{N}, X \in (\mathcal{L}_E(H))_{n,n},$ and $\zeta \in \hat{E}'$,

$$\frac{1}{\sqrt{n}} \sum_{j \in \mathbb{N}_n} \left( \sum_{i \in \mathbb{N}_n} \langle (\varphi_{z,t}X_j)\zeta_i | \zeta_j \rangle \right) =$$

$$= \sum_{i,j \in \mathbb{N}_n} \langle X_{ij} u_t \zeta_i | u_t \zeta_j \rangle \geq 0$$

([1] Theorem 5.6.6.1 f) and [1] Theorem 5.6.1.11 $c_1 \Rightarrow c_2$) so $\varphi_{z,t}$ is completely positive ([1] Theorem 5.6.6.1 f) and [1] Theorem 5.6.1.11 $c_2 \Rightarrow c_1$).

e) By Proposition 2.1.4 a),d) and Proposition 2.1.3 b),

$$\varphi_{z,t}((x \otimes 1_k)V_r) = u_t^*(x \otimes 1_k)V_ru_t = xu_t^*Vu_t = xu_t^*u_rf(r,t) = \delta_{z,rf}(r,t)x.$$

f) By e) (and Proposition 1.1.2 a)),

$$\varphi_{z,t}X = \sum_{r \geq 1} \varphi_{z,t}((x \otimes 1_k)V_r) = \sum_{r \geq 1} \delta_{z,rf}(r,t)x_r = f(st^{-1}, t)x_{s,t^{-1}},$$

$$x_t = \varphi_{z,t}X = f(t,1)x_t = x_t.$$ 

g) By Proposition 2.1.4 c),d),

$$\varphi_{z,t}((x \otimes 1_k)X(y \otimes 1_k)) = u_t^*(x \otimes 1_k)X(y \otimes 1_k)u_t =$$

$$= xu_t^*Xu_t y = x(\varphi_{z,t}X)y .$$
DEFINITION 2.1.7 We put
\[ R(f) := \left\{ \sum_{t \in T} (x_t \otimes 1_k) V_t \mid (x_t)_{t \in T} \in E(T) \right\}, \]
\[ S(f) := \frac{R(f)}{\| \|}, \quad S_{||f||} := \frac{R(f)}{||f||} \]
and call \( S(f) \) the **Schur product associated to** \( f \). Moreover we put \( S_C(f) := S(f) \) in the C*-case and \( S_W(f) := S(f) \) in the W*-case. If \( F \) is a subset of \( E \) then we put
\[ S(f, F) := \left\{ \sum_{t \in T} (x_t \otimes 1_k) V_t \mid (x_t)_{t \in T} \in E(T) \right\}_{t \in F} \]
and use similar notation for the other \( S \).

By Proposition 2.1.2 b),d),e), \( R(f) \) is an involutive unital \( E \)-subalgebra of \( L_E(H) \) (with \( V_1 \) as unit). In particular \( S_{||f||} \) is an \( E \)-C*-subalgebra of \( L_E(H) \). If \( T \) is finite then \( R(f) = S(f) \). By Corollary 1.3.7 e), \( S_C(f)_{2_3} \) is complete.

PROPOSITION 2.1.8 For \( X \in R(f) \) and \( s, t \in T \),
\[ \phi_{s,t} X = f(st^{-1}, t)X_{st^{-1}}. \]

Let \( \mathcal{R} \) be a filter on \( R(f) \) converging to \( X \) in the \( \mathcal{Z}_1 \)-topology. By Proposition 2.1.6 c),f) (and Corollary 1.3.7 d)),
\[ \phi_{s,t} X = \lim_{Y \to \mathcal{R} \mathcal{R}} \phi_{s,t} Y = \lim_{Y \to \mathcal{R} \mathcal{R}} f(st^{-1}, t)Y_{st^{-1}} = f(st^{-1}, t) \lim_{Y \to \mathcal{R} \mathcal{R}} Y_{st^{-1}} = f(st^{-1}, t) \lim_{Y \to \mathcal{R} \mathcal{R}} Y_{st^{-1}}X = f(st^{-1}, t)X_{st^{-1}}. \]

THEOREM 2.1.9 Let \( X \in R(f) \).

a) If \( (x_t)_{t \in T} \) is a family in \( E \) such that
\[ X = \sum_{t \in T} (x_t \otimes 1_k) V_t \]
then \( X_t = x_t \) for every \( t \in T \). In particular, if \( T \) is finite then the map
\[ E^T \to S(f), \quad x \mapsto \sum_{t \in T} (x_t \otimes 1_k) V_t \]
is bijective and \( E \)-linear (Proposition 2.1.2 d)).

b) We have
\[ X = \sum_{t \in T} (x_t \otimes 1_k) V_t \in S(f) \]
c) \( (X^*)_t = \tilde{f}(t)(X_t)^* \) for every \( t \in T \) and
\[ X^* = \sum_{t \in T} (X_t^* \otimes 1_k) V_t^* \in R(f) \]
d) \( S(f) = \frac{R(f)}{||f||} = \frac{R(f)}{\| f \|} \).
e) For \( \xi \in H \) and \( t \in T \),
\[ (X\xi)_t = \sum_{s \in T} f(s, s^{-1}t)X_s \xi_{s^{-1}t} \].
f) If $T$ is finite and if we identify $\mathcal{L}_E(H)$ with $E_{\pi T}$ then $X$ is identified with the matrix

$$[f(st^{-1}, t)X_{st^{-1}}]_{s,t \in T},$$

and for every $r \in T$, $V_r$ is identified with the matrix

$$[f(st^{-1}, t)\delta_{s,r}]_{s,t \in T}.$$

g) If $X, Y \in S(f)$ and $t \in T$ then $XY \in S(f)$ and

$$(XY)_t = \sum_{s \in T} f(s, t)x_sy_s,$$

$$(X^*Y)_t = \sum_{s \in T} f(t, s)x^*_sy_s,$$

$$(X^*Y)_1 = \sum_{s \in T} x^*_s,$$

$$(XY^*)_1 = \sum_{s \in T} x_s.$$

h) The map

$$E \rightarrow S(f), \quad x \mapsto x \otimes 1_K$$

is an injective unital $C^*$-homomorphism and so $S(f)$ is an $E$-$C^*$-subalgebra of $\mathcal{L}_E(H)$ and $Re S(f)$ is closed in $S(f)_1$. In the $W^*$-case, $S_W(f)$ is the $W^*$-subalgebra of $\mathcal{L}_E(H)$ generated by $R(f)$ and $R(f)^\#$ is dense in $S_W(f)^\# = S_W(f)_{s^2}$, which is compact.

i) If $E$ is a $W^*$-algebra then $S_C(f)$ may be identified canonically with a unital $C^*$-subalgebra of $S_W(f)$ by using the map of Proposition 1.3.9 b). By this identification $S_C(f)$ generates $S_W(f)$ as $W^*$-algebra.

j) If $F$ is a closed ideal of $E$ (resp. of $E_E$) then $S(f, F)$ is a closed ideal of $S(f)$ (resp. of $S(f)$).

k) If $F$ is a unital $C^*$-subalgebra of $E$ such that $f(s, t) \in F$ for all $s, t \in T$ then $S(f, F)$ is a unital $C^*$-subalgebra of $S(f)$ and the map

$$S(f, F) \rightarrow S(g), \quad X \mapsto \sum_{t \in T} (x_t \otimes 1_K)V_t^\theta$$

is an injective $C^*$-homomorphism, where

$$g : T \times T \rightarrow Un, \quad (s, t) \mapsto f(s, t).$$

This map induces a $C^*$-isomorphism $S_{\perp f}(f, F) \rightarrow S_{\perp g}(g)$.

l) $(X, Y) \in S(f)_1 \Rightarrow (x_t, y_t) \in E_x$

a) By Proposition 2.1.6 c), e),

$$x_t = \varphi_{x_t}X = \sum_{s \in T} \varphi_{x_t}((x_s \otimes 1_K)V_t) = \sum_{s \in T} \delta_{s,t}f(s, 1)x_s = x_t.$$

b&c&d

Step 1 \hspace{1cm} $X = \sum_{t \in T} (x_t \otimes 1_K)V_t$
By Proposition 2.1.3 d), Corollary 1.3.7 d), Proposition 2.1.8, and Proposition 2.1.4 b), d),
\[
X = \left( \sum_{s \in T} u_s^* u_t^* \right) \left( X \left( \sum_{s \in T} u_s u_t^* \right) = \sum_{s \in T} u_s^* u_t^* X u_s u_t^* = \sum_{s \in T} u_s \phi_{s,t}(X) u_t^* = \sum_{s \in T} u_s f(st^{-1}, t) X_{s, t} u_t^* = \sum_{s \in T} u_s X_s f(r, r^{-1}s) u_t^* = \sum_{s \in T} u_s X_s V_r u_t^* = \sum_{s \in T} X_s \otimes 1_K V_r \right)
\]

Step 2 b&c&d
By Step 1, Corollary 1.3.7 a), and Proposition 2.1.2 d,e) (and Proposition 1.1.2 a)),
\[
X^* = \left( \sum_{s \in T} (X_s \otimes 1_K) V_r \right)^* = \sum_{s \in T} (X_s \otimes 1_K)^* V_r = \sum_{s \in T} (X_s \otimes 1_K)^* (\tilde{f}(s) \otimes 1_K) V_{r^{-1}} = \sum_{s \in T} (\tilde{f}(r) X_{r^{-1}})^* \otimes 1_K V_r \in \mathcal{K}(f).
\]

By a),
\[
(X^*)_r = \tilde{f}(t)(X_{r^{-1}})^*.
\]

By Step 1 and Proposition 2.1.2 e) (and Proposition 1.1.2 a)),
\[
X^* = \sum_{t \in T} ((X^*)_t \otimes 1_K) V_t = \sum_{t \in T} (X_{t^{-1}})^* \otimes 1_K (\tilde{f}(t) \otimes 1_K) V_t = \sum_{t \in T} (X_{t^{-1}})^* \otimes 1_K V_{t^{-1}} = \sum_{t \in T} (X_{t^{-1}})^* \otimes 1_K V_{t^{-1}}^*.
\]

Together with Step 1 this proves
\[
X = \sum_{t \in T} (X_t \otimes 1_K) V_t \in S(f), \quad X^* = \sum_{t \in T} (X_t)^* \otimes 1_K V_t^* \in S(f).
\]

In particular \( S(f) = \frac{T}{\mathcal{K}(f)} = \frac{T}{\mathcal{K}(f)} \).

e) By b) and Corollary 1.3.7 b), in the C*-case,
\[
\langle X \xi \rangle_r = \left\langle \sum_{s \in T} (X_s \otimes 1_K) V_r \xi \mid 1_{E} \otimes e_t \right\rangle = \sum_{s \in T} \langle X_s \otimes 1_K V_r \xi \mid 1_{E} \otimes e_t \rangle = \sum_{s \in T} X_s f(s, s^{-1}t) \xi_{s^{-1}t} = \sum_{s \in T} f(s, s^{-1}t) X_\xi_{s^{-1}t}.
\]

The proof is similar in the W*-case.

f) For \( \xi \in H \) and \( s \in T \), by e),
\[
\langle X \xi \rangle_r = \sum_{t \in T} f(t, t^{-1}s) X_\xi_{s^{-1}t} = \sum_{t \in T} f(st^{-1}, r) X_\xi_{r^{-1}t}.
\]
g) By b), Corollary 1.3.7 b,d), and Proposition 2.1.2 b,d),

\[ XY = \sum_{s \in T} (X_s \otimes 1_k)V_s \sum_{t \in T} (X_t \otimes 1_k)V_t = \]

\[ = \sum_{s \in T} \sum_{t \in T} (X_s \otimes 1_k)(Y_t \otimes 1_k)V_s V_t = \]

\[ = \sum_{s \in T} \sum_{t \in T} (f(s, t) \otimes 1_k)V_s V_t = \]

\[ = \sum_{s \in T} \sum_{t \in T} (f(s, s^{-1}r)X_s Y_{s^{-1}r}) \otimes 1_k)V_s. \]

Since by d),

\[ \sum_{s \in T} (f(s, s^{-1}r)X_s Y_{s^{-1}r}) \otimes 1_k)V_s \in \mathbb{S}(f) \]

for every \( s \in T \) we get \( XY \in \mathbb{S}(f) \), again by d). By Corollary 1.3.7 b) and Proposition 2.1.6 c,e),

\[ (XY)_t = \Phi_{s, t}(XY) = \sum_{s \in T} \sum_{r \in T} \Phi_{s, r}(f(s, s^{-1}r)X_s Y_{s^{-1}r}) \otimes 1_k) V_s = \]

\[ = \sum_{s \in T} \sum_{r \in T} \delta_{s, r} f(r, 1)f(s, s^{-1}r)X_s Y_{s^{-1}r} = \sum_{s \in T} f(s, s^{-1}t)X_s Y_{s^{-1}t}. \]

By the above, c), and Proposition 1.1.2 b),

\[ (X^* Y)_t = \sum_{s \in T} f(s, s^{-1}t)(X^*_s) Y_{s^{-1}t} = \sum_{s \in T} f(s, s^{-1}t)\bar{f}(s)(X_s^*)^* Y_{s^{-1}t} = \]

\[ = \sum_{s \in T} f(s^{-1}, t)\bar{X}_s Y_{s^{-1}t} = \sum_{r \in T} f(r, t)Y_r X_{r^{-1}t}, \]

\[ (XY^*)_t = \sum_{s \in T} f(s, s^{-1}t)X_s (Y^*_t) = \sum_{s \in T} f(s, s^{-1}t)Y_r^* \bar{f}(s^{-1}t)(Y_t^*)^* = \]

\[ = \sum_{r \in T} f(t, t^{-1}s)X_r (Y^*_{t^{-1}s})^* = \sum_{s \in T} f(t, s)^*X_s Y^*_t. \]

It follows by Proposition 1.1.2 a),

\[ (X^* Y)_t = \sum_{s \in T} X_s^* Y_s, \quad (XY^*)_t = \sum_{s \in T} X_s Y_s^*. \]

h) By c) and g), \( \mathbb{S}(f) \) is an involutive unital subalgebra of \( \mathbb{L}(H) \). Being closed (resp. closed in \( \mathbb{L}(H) \) d) and Corollary 1.3.7 c))) it is a \( C^* \)-subalgebra of \( \mathbb{L}(H) \) (resp. generated by \( \mathcal{R}(f) \) b) and Corollary 4.4.12 a) and by Corollary 6.3.8.7 \( \mathbb{R}(f)^* \) is dense in \( \mathcal{S}(f)^* \), which is compact by Corollary 1.3.7 c)). The assertion concerning \( E \) follows from Proposition 2.1.2 d) and Lemma 1.3.2 c). By Corollary 1.3.7 a), \( \mathbb{S}(f) \) is a closed set of \( \mathbb{S}(f)^*_2 \).

i) The assertion follows from h), Proposition 1.3.9 b), and Lemma 1.3.8 c) \( \Rightarrow a \).

j) For \( X \in \mathbb{S}(f), \ Y \in \mathbb{S}(f), \) and \( t \in T \), by g), \( (XY)_t, (YX)_t \in \mathbb{S}(f, F) \) so \( \mathbb{S}(f, F) \) is an ideal of \( \mathbb{S}(f) \). The closure properties follow from Proposition 2.1.6 c).

k) By c) and g), \( \mathbb{S}(f, F) \) is a unital involutive subalgebra of \( \mathbb{S}(f) \) and by Proposition 2.1.6 c), \( \mathbb{S}(f, F) \) is a \( C^* \)-subalgebra of \( \mathbb{S}(f) \). The last assertion follows from the fact that the image of the map contains \( \mathcal{R}(g) \).
I) There are $U, V \in S(f)$ with


For $t \in T$,

$$0 \leq (U_t, V_t)^* (U_t, V_t) = (U_t^*, -V_t^*)(U_t, V_t) = (U_t^* U_t + V_t^* V_t, U_t^* V_t - V_t^* U_t).$$

By g),

$$X_1 = (U_1^* U + V_1^* V) = \sum_{t \in T} (U_t^* U_t + V_t^* V_t),$$
$$Y_1 = (U_1^* V - V_1^* U) = \sum_{t \in T} (U_t^* V_t - V_t^* U_t),$$

so

$$(X_1, Y_1) = \sum_{t \in T} (U_t^* U_t + V_t^* V_t, U_t^* V_t - V_t^* U_t) \in E_1^0.$$  

\[ \blacksquare \]

Remark. It may happen that by the identification of i), $S_c(f) \neq S_W(f)$ (Remark of Proposition 2.1.23).

**COROLLARY 2.1.10**

a) If $(x_t)_{t \in T}$ is a family in $E$ such that $(\|x_t\|)_{t \in T}$ is summable then

$$(x_t \otimes 1_k)_{t \in T}$$

is norm summable in $L_E(H)$ and

$$\left\| \sum_{t \in T} (x_t \otimes 1_k) V_t \right\| = \sum_{t \in T} \|x_t\|.$$

b) The set

$$A = \left\{ X \in S(f) \mid \sum_{t \in T} \|X_t\| < \infty \right\}$$

is a dense involutive unital subalgebra of $S_{\|\cdot\|}(f)$ with

$$\sum_{t \in T} \|(X_t)\| = \sum_{t \in T} \|X_t\|,$$
$$\sum_{t \in T} \|(X Y)_t\| \leq \left( \sum_{t \in T} \|X_t\| \right) \left( \sum_{t \in T} \|Y_t\| \right)$$

for all $X, Y \in A$.

c) $A$ endowed with the norm

$$A \rightarrow \mathbb{R}_+^+, \quad X \rightarrow \sum_{t \in T} \|X_t\|$$

is an involutive Banach algebra and $S_{\|\cdot\|}(f)$ is its $C^*$-hull.
a) For $S \in \mathfrak{S}_f(T)$, by Proposition 2.1.2 e),
\[
\left\| \sum_{t \in T} (x_t \otimes 1_k) V_t \right\| \leq \sum_{t \in T} \| x_t \otimes 1_k \| \| V_t \| = \sum_{t \in T} \| x_t \|
\]
and the assertion follows.

b) By Theorem 2.1.9 c), $X^* \in S(f)$ and
\[
\| (X^*)_t \| = \| (X_{t-1})^* \| = \| X_{t-1} \|
\]
for all $t \in T$ so
\[
\sum_{t \in T} \| (X^*)_t \| = \sum_{t \in T} \| X_{t-1} \| = \sum_{t \in T} \| X_t \|.
\]
By Theorem 2.1.9 g), $XY \in S(f)$ and
\[
\| (XY)_t \| = \left\| \sum_{s \in T} f(s, s^{-1} t) X_s Y_{s^{-1}} \right\| \leq \sum_{s \in T} \| X_s \| \| Y_{s^{-1}} \|
\]
for every $t \in T$ so
\[
\sum_{t \in T} \| (XY)_t \| \leq \sum_{t \in T} \sum_{s \in T} \| X_s \| \| Y_{s^{-1}} \| = \sum_{t \in T} \| X_t \| \left( \sum_{s \in T} \| Y_{s^{-1}} \| \right) = \\
= \sum_{t \in T} \| X_t \| \left( \sum_{s \in T} \| Y_s \| \right) = \left( \sum_{t \in T} \| X_t \| \right) \left( \sum_{t \in T} \| Y_t \| \right).
\]

c) is easy to see. \hfill \blacksquare

Remark. There may exist $X \in \mathfrak{S}_f$ for which $((x_t \otimes 1_k) V_t)_{t \in T}$ is not norm summable, as it is known from the theory of trigonometric series (see Proposition 819). In particular the inclusion $A \subset \mathfrak{S}_f$ may be strict.

**COROLLARY 2.1.11** Let $F$ be a unital $C^*$-algebra and $\tau : E \to F$ a positive continuous (resp. $W^*$-continuous) unital trace.

a) $\tau \circ \varphi_{1,1}$ is a positive continuous (resp. $W^*$-continuous) unital trace.

b) If $\tau$ is faithful then $\tau \circ \varphi_{1,1}$ is faithful and $V_1$ is finite.

c) In the $W^*$-case, $S\tau(f)$ is finite iff $E$ is finite.

a) Let $X, Y \in S(f)$. By Theorem 2.1.9 g) (and Proposition 1.1.2 a)),
\[
\tau \varphi_{1,1}(XY) = \tau \left( \sum_{t \in T} f(t, t^{-1}) X_t Y_{t^{-1}} \right) = \tau \left( \sum_{t \in T} f(t, t^{-1}) X_{t-1} Y_t \right) = \\
= \sum_{t \in T} \tau (f(t, t^{-1}) X_{t-1} Y_t) = \sum_{t \in T} \tau (f(t, t^{-1}) Y_{t-1} X_t) = \tau \left( \sum_{t \in T} f(t, t^{-1}) Y_{t-1} X_t \right) = \\
= \tau \varphi_{1,1}(YX).
\]
Thus $\tau \circ \varphi_{1,1}$ is a trace which is obviously positive, continuous (resp. $W^*$-continuous), and unital (Proposition 2.1.6 c),d)).

b) By Theorem 2.1.9 g), $\varphi_{1,1}$ is faithful, so $\tau \circ \varphi$ is also faithful. Let $X \in S(f)$ with $X^*X = V_1$. By a),
\[
\tau \varphi_{1,1}(XX^*) = \tau \circ \varphi_{1,1}(X^*X) = \tau \varphi_{1,1} V_1 = 1_F
\]
so
\[
\tau \varphi_{1,1}(V_1 - XX^*) = 1_F - 1_F = 0, \quad V_1 = XX^*,
\]
and $V_1$ is finite.
c) By b), if \( E \) is finite then \( S_W(f) \) is also finite. The reverse implication follows from the fact that \( E \otimes 1_K \) is a unital \( W^* \)-subalgebra of \( S_W(f) \) (Theorem 2.1.9 h)).

**COROLLARY 2.1.12** Assume \( T \) finite and for every \( x' \in (E')^T \) put
\[
\tilde{x} : S(f) \rightarrow \mathbb{K}, \quad X \rightarrow \sum_{t \in T} (X_t, x'_t).
\]

a) \( \tilde{x} \in S(f)' \) and
\[
\sup_{t \in T} \|x'_t\| \leq \|\tilde{x}\| \leq \sum_{t \in T} \|x'_t\|
\]
for every \( x' \in (E')^T \) and the map
\[
\varphi : (E')^T \rightarrow S(f)', \quad x' \mapsto \tilde{x}
\]
is an isomorphism of involutive vector spaces such that
\[
\varphi(x'x) = (x \otimes 1_K)(\varphi x'), \quad \varphi(x'x) = (\varphi x')(x \otimes 1_K)
\]
([1] Proposition 2.2.7.2) for every \( x \in E \) and \( x' \in (E')^T \).

b) If \( E \) is a \( W^* \)-algebra then the map
\[
\psi : (E)^T \rightarrow S(f)', \quad (a_t)_{t \in T} \mapsto (\tilde{a_t})_{t \in T}
\]
is an isomorphism of involutive vector spaces such that
\[
\psi(xa) = (x \otimes 1_K)(\psi a), \quad \psi(ax) = (\psi a)(x \otimes 1_K)
\]
for every \( x \in E \) and \( a \in (E)^T \).

**COROLLARY 2.1.13** Assume \( T \) finite and let \( M \) be a Hilbert right \( S(f) \)-module. \( M \) endowed with the right multiplication
\[
M \times E \rightarrow M, \quad (\xi, x) \mapsto (x \otimes 1_K)
\]
and with the inner-product
\[
M \times M \rightarrow E, \quad (\xi, \eta) \mapsto (\xi \eta)_1
\]
is a Hilbert right \( E \)-module denoted by \( \tilde{M} \), \( L_{S(f)}(M) \) is a unital \( C^* \)-subalgebra of \( L_E(\tilde{M}) \), and \( M \) is selfdual if \( \tilde{M} \) is so.

By Proposition 2.1.6 d),g) and Theorem 2.1.9 g),l), for \( X, Y \in S(f) \) and \( x \in E \),
\[
\phi_{11}(X(x \otimes 1_K)) = (\phi_{11}X)x, \quad X \geq 0 \Rightarrow \phi_{11}X \geq 0,
\]
\[
(X, Y) \in S(f)_+ \Rightarrow (\phi_{11}X, \phi_{11}Y) \in E_+^*, \quad \inf\{\|\phi_{11}X\| \mid X \in S(f)_+, \|X\| = 1\} > 0
\]
and the assertion follows from Proposition 2.1.6 a),c),d) and [1] Proposition 5.6.2.5 a),c),d).

**COROLLARY 2.1.14** Let \( n \in \mathbb{N} \) and let \( \varphi : S(f) \rightarrow E_{nK} \) be an \( E-C^* \)-homomorphism. Then \( \varphi V_{ij} \in E' \) for all \( t \in T \) and all \( i, j \in \mathbb{N}_n \).

For \( x \in E \), by Proposition 2.1.2 d) and Theorem 2.1.9 h),
\[
x(\varphi V_t) = \varphi(x \otimes 1_K)(\varphi V_t) = \varphi((x \otimes 1_K)V_t) = \varphi(V_t(x \otimes 1_K)) = (\varphi V_t)\varphi(x \otimes 1_K) = (\varphi V_t)x
\]
so \( (\varphi V_t)_{ij} \in E' \).
COROLLARY 2.1.15 Let $S$ be a group and $g \in \mathcal{F}(S, S(f))$. If we put

$$h : (T \times S) \times (T \times S) \rightarrow \text{Un} \quad S(f)^{\gamma},$$

then $h \in \mathcal{F}(T \times S, S(f))$.

The assertion follows from Theorem 2.1.9 h).

COROLLARY 2.1.16 Let $X \in S(f)$ (resp. $X \in S_{||f||}(f)$).

a) For every $S \in T$,

$$\sum_{s \in S} (X \otimes 1_{K}) V_{s} \in S(f) \quad \text{(resp. } \sum_{s \in S} (X \otimes 1_{K}) V_{s} \in S_{||f||}(f)\text{)}$$

and

$$\gamma := \sup \left\{ \left\| \sum_{s \in S} (X \otimes 1_{K}) V_{s} \right\| : S \in \mathcal{B}_{f}(T) \right\} < \infty.$$

b) We put for every $\alpha \in l^{\infty}(T)$

$$aX : T \rightarrow E, \quad t \mapsto a_{t}X_{t}.$$

Then $aX \in S(f)$ (resp. $aX \in S_{||f||}(f)$) for every $\alpha \in l^{\infty}(T)$ and the map

$$l^{\infty}(T) \rightarrow S(f) \quad \text{(resp. } S_{||f||}(f)),$$

is norm-continuous.

c) Assume $E$ is a $W^{*}$-algebra and let $l^{\infty}(T, E)$ be the $C^{*}$-direct product of the family $\{E_{t} : t \in \gamma\}$, which is a $W^{*}$-algebra ([1] Proposition 4.4.4.21 a)). We put for every $\alpha \in l^{\infty}(T, E)$,

$$aX : T \rightarrow E, \quad t \mapsto a_{t}X_{t}.$$

Then $aX \in S_{W}(f)$ for every $\alpha \in l^{\infty}(T, E)$ and the map

$$l^{\infty}(T, E) \rightarrow S_{W}(f), \quad \alpha \mapsto aX$$

is continuous and $W^{*}$-continuous.

a) In the $C^{*}$-case the family $\{(X_{t} \otimes 1_{K}) V_{s} : s \in S\}$ is summable since $S_{C}(f)_{2s}$ is complete. By Banach-Steinhaus Theorem, $\gamma$ is finite. In the $W^{*}$-case the summability follows now from Corollary 1.3.7 b), c) and Theorem 2.1.9 b).

b) Let $G$ be the vector subspace $\{\alpha \in l^{\infty}(T) \mid \alpha(T) \text{ is finite}\}$ of $l^{\infty}(T)$. By a), the map

$$G \rightarrow S(f) \quad \text{(resp. } S_{||f||}(f)), \quad \alpha \mapsto aX$$

is well-defined, linear, and continuous. The assertion follows by continuity.

c) Let $x \in E, S \subset T$, and $\alpha := x_{E_{T}}$. For $\xi, \eta \in H$ and $a \in E$, by a) and Lemma 1.3.2 b) (and Theorem 2.1.9 b)),

$$\left\langle \alpha X, (a, \xi, \eta) \right\rangle = \left\langle (\alpha X) \xi, \eta \right\rangle = \sum_{t \in T} \eta_{t} x((e_{t}X) \xi)_{t}, a \right\rangle =$$

$$= \sum_{t \in T} \left\langle x, (e_{t}X) \xi, a_{t} \eta_{t} \right\rangle.$$

Let $G$ be the involutive subalgebra $\{\alpha \in l^{\infty}(T, E) \mid \alpha(T) \text{ is finite}\}$ of $l^{\infty}(T, E)$ and let $\tilde{G}$ be its norm-closure in $l^{\infty}(T, E)$, which is a $C^{*}$-subalgebra of $l^{\infty}(T, E)$. By [1] Proposition 4.4.4.21 a), $G$ is dense in $l^{\infty}(T, E)_{F}$, where $F := l^{\infty}(T, E)$. 36
Let $\alpha \in L^\infty(T, E)^g$ and let $\mathfrak{F}$ be a filter on $G^a$ converging to $\alpha$ in $L^\infty(T, E)^g$ ([1] Corollary 6.3.8.7). By the above (and by Theorem 2.1.9 h),

$$\lim_{\beta \in \mathfrak{F}} \beta X = \alpha X$$

in $S_W(f) \downarrow\downarrow$ and so $\alpha X \in S_W(f)$. The assertion follows.

COROLLARY 2.1.17 Let $S$ be a subgroup of $T$. Put

$$f_S := f|_{(S \times S)}, \quad K_S := K(S), \quad G := \{X \in S(f) \mid t \in T \setminus S \Rightarrow X_t = 0\}.$$

a) $f_S \in F(S, E)$.

b) $G$ is an involutive unital subalgebra of $S(f)$.

c) For every $X \in G$, the family $((X_t \otimes 1_{K_S})V_{tS})_{t \in S}$ is summable in $L_E(K_S)$ and the map

$$\varphi : G \rightarrow S(f_S), \quad X \mapsto \sum_{t \in S} (X_t \otimes 1_{K_S})V_{tS}$$

is an injective $E$-$C^\ast$-homomorphism.

d) If $X \in G \cap S$ then $\varphi X \in S(f_S)$ and the map

$$G \cap S \rightarrow S(f_S), \quad X \mapsto \varphi X$$

is an $E$-$C^\ast$-isomorphism.

e) If $S$ is finite then the map

$$G \rightarrow S(f_S), \quad X \mapsto \sum_{t \in S} (X_t \otimes 1_{K_S})V_{tS}$$

is an $E$-$C^\ast$-isomorphism.

a) is obvious.
b) By Theorem 2.1.9 c), g), $G$ is an involutive unital subalgebra of $S(f)$ and by Proposition 2.1.6 a) (resp. Proposition 2.1.6 c) and Corollary 1.3.7 c)) and Theorem 2.1.9 h), it is an $E$-$C^\ast$-subalgebra of $S(f)$.
c) follows from Theorem 2.1.9 b) and Corollary 2.1.16 a).
d) follows from c).
e) is contained in d).

DEFINITION 2.1.18 We denote by $\Xi_T$ the set of finite subgroups of $T$ and call $T$ **locally finite** if $\Xi_T$ is upward directed and

$$\bigcup_{S \in \Xi_T} S = T.$$ 

$T$ is locally finite iff the subgroups of $T$ generated by finite subsets of $T$ are finite.

COROLLARY 2.1.19 Assume $T$ locally finite. We put $f_S := f|_{(S \times S)}$ for every $S \in \Xi_T$ and identify $S(f_S)$ with $\{X \in S(f) \mid t \in T \setminus S \Rightarrow X_t = 0\}$ (Corollary 2.1.17 e)).

a) For every $X \in S(f_S)$ and $\varepsilon > 0$ there is an $S \in \Xi_T$ such that

$$\left\| \sum_{t \in R} (X_t \otimes 1_{K_S})V_t - X \right\| < \varepsilon$$

for every $R \in \Xi_T$ with $S \subseteq R$.

b) $S(f_S)$ is the norm closure of $\bigcup_{S \in \Xi_T} S(f_S)$ and so it is canonically isomorphic to the inductive limit of the inductive system $

\{S(f_S) \mid S \in \Xi_T\}$ and for every $S \in \Xi_T$ the inclusion map $S(f_S) \rightarrow S(f_S)$ is the associated canonical morphism.
a) There is a $Y \in R(f)$ with $\|X-Y\| < \frac{\varepsilon}{2}$. Let $S \in \mathcal{S}$ with $Y \in S(f)$. By Corollary 2.1.17 b), for $R \in S$ with $S \subset R$,
\[
\left\| \sum_{i \in R} ((X_i-Y_i) \otimes 1_k) V_i \right\| \leq \|X-Y\| < \frac{\varepsilon}{2}
\]
so
\[
\left\| \sum_{i \in R} (X_i \otimes 1_k) V_i - X \right\| \leq \left\| \sum_{i \in R} ((X_i-Y_i) \otimes 1_k) V_i \right\| + \|Y-X\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

b) follows from a).

Remark. The C*-algebras of the form $S_E \|f\|$ with $T$ locally finite can be seen as a kind of AF-E-C*-algebras.

**Proposition 2.1.20** The following are equivalent for all $t \in T$ with $t^2 = 1$ and $\alpha \in \text{Un} E$.

a) $\frac{1}{2} (V_1 + (\alpha \otimes 1_k) V_1) \in \text{Pr} S(f)$.

b) $\alpha^2 = \bar{f}(t)$.

By Proposition 2.1.2 b), d), e),
\[
(V_i)^* = (\bar{f}(t) \otimes 1_k) V_t, \quad (V_i)^2 = (\bar{f}(t)^* \otimes 1_k) V_1
\]
so
\[
\frac{1}{2} (V_1 + (\alpha \otimes 1_k) V_1)^* = \frac{1}{2} (V_1 + ((\alpha \bar{f}(t)) \otimes 1_k) V_t)
\]

\[
\left( \frac{1}{2} (V_1 + (\alpha \otimes 1_k) V_1) \right)^2 = \frac{1}{4} (1_E + \alpha^2 \bar{f}(t)^* \otimes 1_k) V_1 + \frac{1}{2} (\alpha \otimes 1_k) V_1
\]

Thus a) is equivalent to $\alpha \bar{f}(t) = \alpha$ and $\alpha^2 \bar{f}(t)^* = 1_E$, which is equivalent to b).

**Corollary 2.1.21** Let $t \in T$ such that $t^2 = 1$ and $\bar{f}(t) = 1_E$. Then
\[
\frac{1}{2} (V_1 \pm V_t) \in \text{Pr} S(f), \quad (V_1 + V_t)(V_1 - V_t) = 0.
\]
The assertion follows from Proposition 2.1.20.

**Corollary 2.1.22** Let $a, \beta \in \text{Un} E$, $s, t \in T$ with $s^2 = t^2 = 1$, $st = ts$,
\[
\gamma = \frac{1}{2} (\alpha \bar{f}(s, st)^* + \beta \bar{f}(t, st)^*), \quad \gamma' = \frac{1}{2} (\alpha \bar{f}(st, t)^* + \beta \bar{f}(st, s)^*),
\]
and
\[
X = \frac{1}{2} (\alpha \otimes 1_k) V_s + (\beta \otimes 1_k) V_t
\]

a) $f(s, st)f(t, st) = f(st, t)f(st, s) = \bar{f}(st)^*$.

b) $f(s, t)f(s, st) = f(st, s)f(t, st)$.

c) $X^* X = \frac{1}{2} (V_1 + (\gamma \otimes 1_k) V_s)$. \quad \gamma^* = \frac{1}{2} (\alpha \bar{f}(st, t)^* + \beta \bar{f}(st, s)^*)$.

d) The following are equivalent.

\begin{enumerate}
\item $X^* X \in \text{Pr} S(f)$.
\item $XX^* \in \text{Pr} S(f)$.
\item $\alpha \bar{f}(t, st) = \beta \bar{f}(s, st)$.
\item $\alpha \bar{f}(s, st) = \beta \bar{f}(s, st)$.
\end{enumerate}
a) and b) follow from the equation of Schur functions (Definition 703 and Proposition 1.1.2 a).

c) By Proposition 2.1.2 b), e) and Proposition 1.1.2 b),

\[ X^* = \frac{1}{2} ((\alpha^* \overline{f}(s)) \otimes 1_E) V_i + ((\beta^* \overline{f}(t)) \otimes 1_E) V_i, \]

\[ X^* X = \frac{1}{2} V_i + \frac{1}{4} ((\alpha^* \beta f(s,t) + \alpha \alpha^* f(t,s)) \otimes 1_E) V_i = \]

\[ = \frac{1}{2} V_i + \frac{1}{4} ((\alpha^* \beta f(s,t)) + \alpha \alpha^* f(t,s)) \otimes 1_E) V_i = \frac{1}{2} (V_i + (\gamma \otimes 1_E) V_i), \]

\[ \alpha^* \beta f(s,t) + \alpha \alpha^* f(t,s) \otimes 1_E) V_i = \frac{1}{2} (V_i + (\gamma \otimes 1_E) V_i). \]

\[ d_1 \iff d_2 \text{ is known.} \]

\[ d_1 \iff d_3. \text{ By a),} \]

\[ \gamma^2 - \overline{f}(st) = \frac{1}{4} (\alpha^* \beta \beta f(s,t) - \beta \alpha^* af(t,s))^2 + 2f(s,t)^* f(t,st) - \]

\[ f(s,t)^* f(t,st) = \frac{1}{4} (\alpha^* \beta f(s,t) - \beta \alpha^* af(t,s))^2. \]

By Proposition 2.1.20 d_1 is equivalent to \( \gamma^2 = \overline{f}(st) \) so, by the above, since \( \alpha^* \beta f(s,t) - \beta \alpha^* af(t,s) \) is normal, it is equivalent to \( \alpha^* \beta f(s,t) = \beta \alpha^* af(t,s) \) or to \( \beta \alpha^* af(s,t) = \alpha^* \beta f(t,s). \)

\[ d_3 \iff d_4 \text{ follows from b).} \]

**PROPOSITION 2.1.23** Let \( X \in S(f). \)

\[ a) \sum_{t \in T} X_t X_t = (X^* X)_1, \sum_{t \in T} (X_t X_t^*) = (XX^*)_1, \]

\[ b) (X_t)_{t \in T}, (X_t^*)_{t \in T} \in \overline{\mathbb{C}} \overline{E}, \]

\[ \| (X_t)_{t \in T} \| \leq \| X \|, \quad \| (X_t^*)_{t \in T} \| \leq \| X \|. \]

\[ c) \text{ If } T \text{ is finite and } f \text{ is constant then there is an } X \in S(f) \text{ with} \]

\[ \| X \| \geq \sqrt{\text{Card} T} \| (X_t)_{t \in T} \|, \quad \| X \| \geq \sqrt{\text{Card} T} \| (X_t^*)_{t \in T} \|. \]

\[ d) \text{ If } T \text{ is infinite and locally finite and } f \text{ is constant then the map} \]

\[ S(f) \rightarrow \overline{\mathbb{C}} \overline{E}, \quad X \mapsto (X_t)_{t \in T} \]

is not surjective.

a) follows from Theorem 2.1.9 g).

b) By a),

\[ (X_t)_{t \in T}, (X_t^*)_{t \in T} \in \overline{\mathbb{C}} \overline{E} \]

and by Proposition 2.1.6 a),

\[ \| (X_t)_{t \in T} \|^2 = \| (X^* X)_{t \in T} \| \leq \| X^* X \| = \| X \|^2, \]

\[ \| (X_t^*)_{t \in T} \|^2 = \| (X X^*)_{t \in T} \| \leq \| XX^* \| = \| X \|^2. \]

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c) Let \( n \) := \( \text{Card} \ T \) and for every \( t \in T \) put \( X_t := I_{E_t}, \xi_t := 1_{E_t} \). Then

\[
\| (X_t)_{t \in T} \|^2 = \| (X_t^*)_{t \in T} \|^2 = n, \quad \| (\xi_t)_{t \in T} \|^2 = n.
\]

For \( t \in T \), by Theorem 2.1.9 e),

\[
(X_t) = \sum_{s \in T} f(s, s^{-1} t) X_s \xi_{s^{-1}} = n 1_{E_t}
\]

so

\[
\langle X \xi, X \xi \rangle = n^2 1_{E_t}, \quad n \| X \|^2 = \| X \|^2 \| \xi \|^2 = n^3,
\]

\[
\| X \|^2 \geq n \| (X_t)_{t \in T} \|^2, \quad \| X \|^2 \geq n \| (\xi_t)_{t \in T} \|^2.
\]

d) follows from c), Theorem 2.1.9 a), and the Principle of Inverse Operator.

Remark. If \( E \) is a W*-algebra then it may exist a family \( \{X_t\}_{t \in T} \in E \) such that the family \( \{ (x_t \otimes 1_K)_{t \in T} \} \) is summable in \( L_E(H)_{2_1} \) in the W*-case but not in the C*-case as the following example shows. Take \( T = \mathbb{Z}, f = f(2), E = \mathbb{F}^N(\mathbb{Z}), \) and \( x_t = \{\delta_{t, n}\}_{n \in T} \). By Proposition 2.1.23 b), \( \{ (x_t \otimes 1_K)_{t \in T} \} \) is not summable in \( L_E(H)_{2_1} \) in the C*-case. In the W*-case for \( \xi \in H \) and \( s, t \in T \),

\[
\langle ( (x_t \otimes 1_K)_{t \in T} \xi), ( (x_t \otimes 1_K)_{t \in T} \xi) \rangle = \xi_n |\xi_n|^2,
\]

\[
\langle (x_t \otimes 1_K)_{t \in T} \xi, (x_t \otimes 1_K)_{t \in T} \xi \rangle = e_t |\xi|^2.
\]

Thus

\[
X = \sum_{t \in T} (x_t \otimes 1_K)_{t \in S_W(f)}.
\]

Using the identification of Theorem 2.1.9 i), we get \( X \in \text{S}(f) \setminus \text{S}(f) \).

**COROLLARY 2.1.24** Let \( X \in \text{S}(f) \).

a) \( X \in \{ x \otimes 1_K \mid x \in E \} \) iff \( X_t \in E_t \) for all \( t \in T \).

b) \( X \in \{ V_t \mid t \in T \} \) iff

\[
X_{ts^{-1}} = f(s, s^{-1} ts)^* f(t, s) X_t = f(s^{-1}, ts) f(t, s) f(s)^* f(t, s) X_t
\]

for all \( s, t \in T \).

c) \( X \in \text{S}(f) \) iff for all \( s, t \in T \)

\[
X_t \in E^c, \quad X_{ts} = f(s, s^{-1} ts)^* f(t, s) X_t = f(s^{-1}, ts) f(t, s) f(s)^* f(t, s) X_t.
\]

In particular if \( f(s, t) = f(t, s) \) for all \( s, t \in T \) then \( X \in \text{S}(f) \) iff \( X_t \in E \) for all \( t \in T \).

d) \( \varphi_{1, 1}(S(f)^c) = E^c \).

e) If the conjugacy class of \( t \in T \) (i.e., the set \( \{ s^{-1} ts \mid s \in T \} \)) is infinite and \( X \in \{ V_t \mid t \in T \} \) then \( X_t = 0 \).

f) If the conjugacy class of every \( t \in T \setminus \{1\} \) is infinite then

\[
\{ V_t \mid t \in T \}^c = \{ x \otimes 1_K \mid x \in E \}, \quad S(f)^c = \{ x \otimes 1_K \mid x \in E^c \}.
\]

Thus if \( E \) is a factor then \( S(f) \) is also a factor.

g) The following are equivalent:

\( g_1 \) \( S(f) \) is commutative.

\( g_2 \) \( T \) and \( E \) are commutative and \( f(s, t) = f(t, s) \) for all \( s, t \in T \).
For \( s, t \in T, x \in E \), and \( Y := (x \otimes 1_k)V_s \) by Theorem 2.1.9 g),

\[
(\alpha Y)_t = \sum_{r \in T} f(r, r^{-1}t)X_rY_{t^{-1}} = \sum_{r \in T} f(r, r^{-1}t)X_r\delta_{r^{-1}t}x = f(ts^{-1}, s|x_{s^{-1}}x,
\]

\[
(\alpha X)_t = \sum_{r \in T} f(r, r^{-1}t)Y_rX_{t^{-1}} = \sum_{r \in T} f(r, r^{-1}t)\delta_{r^{-1}t}xX_{t^{-1}} = f(s, s^{-1}t)xX_{s^{-1}}t.
\]

a) follows from the above by putting \( s = 1 \) (Proposition 1.1.2 a)).

b) follows from the above by putting \( x = 1_E \) and \( t := rs \) (Proposition 1.1.2).

c) follows from a), b), and Corollary 1.3.7 d). The last assertion follows using Proposition 1.1.5 a).

d) follows from c) (and Proposition 1.1.2 a)).

e) follows from b) and Proposition 2.1.23 b).

f) follows from c), e), and Proposition 2.1.2 d).

\[ g_1 \Rightarrow g_2 \] By a), \( E \) is commutative. By Proposition 2.1.2 b),

\[ f(s, t)V_{st} = V_tV_s = V_tV_s = f(t, s)V_tV_s = f(t, s)V_{st} \]

and so by Theorem 2.1.9 a), \( st = ts \) and \( f(s, t) = f(t, s) \).

\[ g_2 \Rightarrow g_1 \] follows from c).

**COROLLARY 2.1.25** If \( \mathbb{K} = \mathbb{R} \) then the following are equivalent:

a) \( S(f)^c = S(f) = \text{Re } S(f) \).

b) \( T \) is commutative, \( E^c = E = \text{Re } E \), and

\[ f(s, t) = f(t, s), \quad \tilde{f}(t) = 1_E, \quad t^2 = 1 \]

for all \( s, t \in T \).

\[ a \Rightarrow b \] By Corollary 2.1.24 \( g_1 \Rightarrow g_2 \); \( T \) is commutative, \( E = E^c \), and \( f(s, t) = f(t, s) \) for all \( s, t \in T \). Since \( E \) is isomorphic with a C*-subalgebra of \( S(f) \) (Theorem 2.1.9 h)), \( E = \text{Re } E \). By Proposition 2.1.2 e),

\[ V_t = V_t^* = (\tilde{f}(t)c 1_k)V_{t^{-1}} \]

so by Theorem 2.1.9 a), \( t = t^{-1} \), \( \tilde{f}(t) = 1_E \), so \( t^2 = 1 \).

\[ b \Rightarrow a \] By Corollary 2.1.24 \( g_2 \Rightarrow g_1 \), \( S(f)^c = S(f) \). For \( X \in S(f) \) and \( t \in T \), by Theorem 2.1.9 c),

\[ (X^*)_t = \tilde{f}(t)(X_{t^{-1}})^* = (X_t)^* = X_t \]

so \( X^* = X \) (Theorem 2.1.9 a)).

**PROPOSITION 2.1.26** Let \( (E_i)_{i \in I} \) be a family of unital C*-algebras such that \( E \) is the C*-direct product of this family. For every \( i \in I \), we identify \( E_i \) with the corresponding closed ideal of \( E \) (resp. of \( E_0 \)) and put

\[ f_i : T \times T \longrightarrow \text{Un } E_i^c, \quad (s, t) \longrightarrow f(s, t)_i. \]

a) For every \( i \in I, f_i \in \mathcal{F}(T, E_i) \). We put (by Theorem 2.1.9 b))

\[ \phi_i : S(f) \rightarrow S(f_i), \quad X \rightarrow \sum_{t \in T} ((X_i) \otimes 1_k)V_t^i. \]

\( \phi_i \) is a surjective C*-homomorphism.
b) In the C*-case, if \( T \) is finite then \( \mathcal{R}(f) = \mathcal{S}_{\|f\|} = \mathcal{S}c(f) \) is isomorphic to the C*-direct product of the family

\[
(\mathcal{R}(f)) = \mathcal{S}_{\|f\|} = \mathcal{S}c(f))_{i\in I}.
\]

c) In the C*-case, if \( I \) is finite then \( \mathcal{S}c(f) \) (resp. \( \mathcal{S}_{\|f\|} \)) is isomorphic to \( \prod_{i\in I} \mathcal{S}c(f_i) \) (resp. \( \prod_{i\in I} \mathcal{S}_{\|f_i\|} \)).

d) In the \( W^* \)-case, \( \mathcal{S}c(f) \) is isomorphic to the C*-direct product of the family \( (\mathcal{S}c(f_i))_{i\in I} \).

Remark. The C*-isomorphisms of b) and c) cease to be surjective in general if \( T \) and \( I \) are both infinite. Take \( T := (\mathbb{Z}_2)^N \), \( I := \mathbb{N} \), \( E_i := \mathbb{K} \) for every \( i \in I \), and \( E := t^o \) (i.e. \( E \) is the C*-direct product of the family \( (E_i)_{i\in I} \)). For every \( n \in \mathbb{N} \) put \( t_n = (\delta_{m,n})_{m\in \mathbb{N}} \in E(T) \). Assume there is an \( X \in \mathcal{S}c(f) \) (resp. \( X \in \mathcal{S}_{\|f\|} \)) with \( \psi X = (V^i)_{i\in I} \) (resp. \( \phi X = (V^i)_{i\in I} \)), where \( \psi \) and \( \phi \) are the maps of b) and c), respectively. Then \( (X_{t_n})_i = \delta_{i,n} \) for all \( i, n \in \mathbb{N} \) and this implies \( (X_{t_n})_{i\in I} \in \mathcal{S}c \), which contradicts Proposition 2.1.23 b).

**PROPOSITION 2.1.27** Let \( S \) be a finite group, \( K := \mathbb{I}(S) \), \( K := \hat{\mathbb{I}}(S \times T) \), and \( g \in \mathcal{F}(S, \mathcal{S}(f)) \) such that \( g(s, s_2) \in \text{Un} \ \mathcal{E} \) (where \( \text{Un} \ \mathcal{E} \) is identified with \( (\text{Un} \ \mathcal{E}) \hat{\otimes} 1_K < \text{Un} \ \mathcal{S}(f) \)) for all \( s, s_2 \in S \) and put

\[
h : (S \times T) \times (S \times T) \rightarrow \text{Un} \ \mathcal{E}, \quad ((s_1, t_1), (s_2, t_2)) \rightarrow g(s_1, s_2) f(t_1, t_2).
\]

a) \( h \in \mathcal{F}(S \times T, E) \); for every \( X \in \mathcal{S}(g) \) put

\[
\phi X := \sum_{i\in I}^{|I|} \left( (X_i)_{t} \otimes 1_K \right) V^i_{\mathcal{E}(S)} \in \mathcal{S}(h).
\]

b) \( \phi : \mathcal{S}(g) \rightarrow \mathcal{S}(h) \) is an E-C*-isomorphism.

a) is obvious.

b) For \( X, Y \in \mathcal{S}(g) \) and \( (s, t) \in S \times T \), by Theorem 2.1.9 c),g) and Proposition 2.1.6 g),

\[
(\phi X^*)_{(s,t)} = ((X^*)_{(s,t)})_{t} = \bar{g}(s)((X_{(s,t)})^*)_{t} = \bar{g}(s) f(t)((X_{(s,t)})^*)_{t} = ((\phi X)^*)_{(s,t)},
\]

\[
(\phi XY)_{(s,t)} = ((XY)_{(s,t)})_{t} = \sum_{r \in S} g(r, r^{-1}s)(X_r Y_{r^{-1}s})_{t} = \sum_{r \in S} \sum_{q \in T} h((r, q), (r, q)^{-1}(s, t)) X_{(r, q)} Y_{(r, q)^{-1}(s, t)} = ((\phi X)(\phi Y))_{(s,t)},
\]

so \( \phi \) is a C*-homomorphism. If \( \phi X = 0 \) then \( X_{(s, t)} = 0 \) for all \( (s, t) \in S \times T \), so \( X = 0 \) and \( \phi \) is injective. Let \( Z \in \mathcal{S}(h) \). For every \( s \in S \) put

\[
X_s := \sum_{i\in I}^{|I|} (Z_{(s, t)} \otimes 1_K) V^i_{\mathcal{E}(S)},
\]

\[
X := \sum_{i\in I} (X_s \otimes 1_K) V^i_{\mathcal{E}(S)}.
\]

Then \( \phi X = Z \) and \( \phi \) is surjective.
PROPOSITION 2.1.28 If $T$ is infinite and $X \in S(f) \setminus \{0\}$ then $X(H^p)$ is not precompact.

Let $t \in T$ with $X_t \neq 0$. There is an $x' \in E_0^+$ (resp. $x' \in E_0^+$) with $(x'_t, x') > 0$. We put $t_1 = 1$ and construct a sequence $(t_n)_{n \in \mathbb{N}}$ recursively in $T$ such that for all $m, n \in \mathbb{N}, m < n$,

$$\left| \langle f(t, t_m)^* f(t_m t_m^{-1}, t_n) X_t X_{t_n t_n^{-1}}, x' \rangle \right| < \frac{1}{2} \langle X_t^* X_t, x' \rangle.$$ 

Let $n \in \mathbb{N} \setminus \{1\}$ and assume the sequence was constructed up to $n - 1$. Since (Proposition 2.1.23 a))

$$\sum_{s \in T} \langle X_{n t_m s}^* X_{n t_m s^{-1}}, x' \rangle < \infty$$

for all $m \in \mathbb{N}_{n-1}$ there is a $t_n \in T$ with

$$\langle X_{n t_m t_n}^* X_{n t_m t_n^{-1}}, x' \rangle < \frac{1}{4} \langle X_t^* X_t, x' \rangle$$

for all $m \in \mathbb{N}_{n-1}$. By Schwarz’ inequality ([1] Proposition 2.3.4.6 c)) for $m \in \mathbb{N}_{n-1}$,

$$\left| \langle f(t, t_m)^* f(t_m t_m^{-1}, t_n) X_t X_{t_n t_n^{-1}}, x' \rangle \right|^2 \leq$$

$$\leq \langle X_t^* X_t, x' \rangle \langle X_{n t_m t_n}^* X_{n t_m t_n^{-1}}, x' \rangle < \frac{1}{4} \langle X_t^* X_t, x' \rangle^2.$$ 

This finishes the recursive construction.

For $r, s \in T$, by Theorem 2.1.9 e),

$$(X(1_E \otimes e_r))_s = \sum_{q \in T} \langle f(q, q^{-1}) s X_d \delta_{q^{-1}, s} = f(s r^{-1}, r) X_{s r^{-1}},$$

$$\langle X(1_E \otimes e_r) X_n \otimes e_n \rangle = f(s r^{-1}, r) X_t^* X_{s r^{-1}}.$$ 

For $m, n \in \mathbb{N}, m < n$, it follows

$$\langle X(1_E \otimes e_m) X_n \otimes e_n \rangle = f(t, t_m) X_t^* X_t,$$

$$\langle X(1_E \otimes e_m) X_n \otimes e_n \rangle = f(t(t_m)^{-1}, t_n) X_t^* X_{t_n t_n^{-1}},$$

$$\langle X(1_E \otimes e_m) X_n \otimes e_n \rangle = f(t(t_m)^{-1}, t_n) X_{t_n t_n^{-1}}.$$ 

Thus the sequence $\left(X(1_E \otimes e_n)\right)_{n \in \mathbb{N}}$ has no Cauchy subsequence and therefore $X(H^p)$ is not precompact.
PROPOSITION 2.1.29 Assume $T$ finite and let $\Omega$ be a compact space, $\omega_0 \in \Omega$,

\[
g : T \times T \to C(\Omega, E), \quad (s, t) \mapsto f(s, t)1_{\Omega},
\]

\[
A = \{X \in S(g) | t \in T, t \neq 1 \Rightarrow X_t(\omega_0) = 0 \},
\]

\[
B = \{Y \in C(\Omega, S(f)) | t \in T, t \neq 1 \Rightarrow Y(\omega_0) = 0 \}.
\]

Then $g \in \mathcal{F}(T, C(\Omega, E))$ and we define for every $X \in A$ and $Y \in B$,

\[
\phi X : \Omega \to S(f), \quad \omega \mapsto \sum_{t \in T} (X_t(\omega) \otimes 1_K) V^f_t,
\]

\[
\psi Y = \sum_{t \in T} (Y(\cdot)_t \otimes 1_K) V^g_t.
\]

Then $A$ (resp. $B$) is a unital $C^*$-subalgebra of $S(g)$ (resp. of $C(\Omega, S(f))$)

\[
\phi : A \to B, \quad \psi : B \to A
\]

are $C^*$-isomorphisms, and $\phi = \psi^{-1}$. It is easy to see that $A$ (resp. $B$) is a unital $C^*$-subalgebra of $S(g)$ (resp. of $C(\Omega, S(f))$) and that $\phi$ and $\psi$ are well-defined. For $X, X' \in A, t \in T, \omega \in \Omega$, by Theorem 2.1.9 c), g) and Proposition 2.1.2 e),

\[
((\phi X)(\phi X'))(\omega)_t = \sum_{s \in T} f(s, s^{-1} t)((\phi X')(\omega))_{s^{-1} t} = \sum_{s \in T} f(s, s^{-1} t)X_{s^{-1} t}(\omega) = (XX')(\omega)_t = (\phi(XX'))(\omega)_t,
\]

\[
(\phi X')(\omega) = \sum_{s \in T} ((X'_s(\omega)) \otimes 1_K) V^f_t = \sum_{s \in T} (f(s)((X'_s(\omega))) \otimes 1_K) V^f_t = \sum_{s \in T} (X_s(\omega)^* \otimes 1_K) (V^f_t)^* = (\phi X')^*(\omega)
\]

so $\phi$ is a $C^*$-homomorphism and we have

\[
(\psi \phi X)_t = \phi X_t = X_t.
\]

Moreover for $Y \in B$,

\[
(\phi \psi Y)_t(\omega) = ((\psi Y)(\omega))_t = Y_t(\omega)
\]

which proves the assertion.

2.2 Variation of the parameters

In this section we examine the changes produced by the replacement of the groups and of the Schur functions.

DEFINITION 2.2.1 We put for every $\lambda \in \Lambda (T, E)$ (Definition 1.1.3)

\[
U_\lambda : H \to H, \quad \xi \mapsto (\lambda(\xi) K_i)_{i \in T}.
\]

It is easy to see that $U_\lambda$ is well-defined, $U_\lambda \in \text{Un}_L(H)$, and the map

\[
\Lambda(T, E) \to \text{Un}_L(H), \quad \lambda \mapsto U_\lambda
\]

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is an injective group homomorphism with \( U_\lambda^* = U_\lambda \) (Proposition 1.1.4 c)). Moreover
\[
\|U_\lambda - U_\mu\| \leq \|\lambda - \mu\|_\infty
\]
for all \( \lambda, \mu \in \Lambda (T, E) \).

**PROPOSITION 2.2.2** Let \( f, g \in \mathcal{F}(T, E) \) and \( \lambda \in \Lambda (T, E) \).

a) The following are equivalent:
   \[ a_1) \; g = f \lambda. \]
   \[ a_2) \; \text{There is a (unique) } E\text{-C}^*\text{-isomorphism} \]
   \[
   \varphi : S(f) \to S(g)
   \]
   continuous with respect to the \( \mathfrak{T}_2 \)-topologies such that for all \( t \in T \) and \( x \in E \),
   \[
   \varphi V_t^f = (\lambda(t) \otimes 1_K) V_t^g
   \]
   (we call such an isomorphism an \( S \)-isomorphism and denote it by \( \approx_S ) \).

b) If the above equivalent assertions are fulfilled then for \( X \in S(f) \) and \( t \in T \),
   \[
   \varphi X = U_\lambda^* X U_\lambda, \quad (\varphi X)_t = \lambda(t)^* X_t.
   \]

c) There is a natural bijection
   \[
   \{S(f)|f \in \mathcal{F}(T, E)\}/\approx_S \to \mathcal{F}(T, E)/\{\delta \lambda | \lambda \in \Lambda (T, E)\}.\]

By Proposition 1.1.4 c), \( \delta \lambda \in \mathcal{F}(T, E) \) for every \( \lambda \in \Lambda (T, E) \).
\( a_1 \Rightarrow a_2 \& b \). For \( s, t \in T \) and \( \xi \in E \), by Proposition 2.1.2 c),
\[
U_s^* V_t^f U_t^s (\xi \otimes e_s) = U_s^* V_t^f (\lambda(s) \xi \otimes e_s) = U_s^* (f(t,s) \lambda(s) \xi) \otimes e_s =
\]
\[
(\lambda(ts)^* f(t,s) \lambda(s) \xi) \otimes e_s = (\lambda(t)^* g(t,s) \xi) \otimes e_s = (\lambda(t)^* \otimes 1_K) V_t^g (\xi \otimes e_s)
\]
so (by Proposition 2.1.2 e))
\[
U_s^* V_t^f U_t^s = (\lambda(t)^* \otimes 1_K) V_t^g.
\]
Thus the map
\[
\varphi : S(f) \to S(g), \quad X \mapsto U_\lambda^* X U_\lambda
\]
is well-defined. It is obvious that it has the properties described in \( a_2 \). The uniqueness follows from Theorem 2.1.9 b).
We have
\[
\varphi((X_t \otimes 1_K)V_t^f) = (X_t \otimes 1_K)(\lambda(t)^* \otimes 1_K)V_t^g = ((\lambda(t)^* X_t) \otimes 1_K)V_t^g
\]
so \( (\varphi X)_t = \lambda(t)^* X_t \).
\( a_2 \Rightarrow a_1 \). Put \( h := \delta \lambda \). By the above, for \( t \in T \),
\[
(\lambda(t)^* \otimes 1_K)V_t^g = \varphi V_t^f = (\lambda(t)^* \otimes 1_K)V_t^h
\]
so \( V_t^f = V_t^h \) and this implies \( g = h \).
\( c \) follows from \( a \).

**Remark.** Not every \( E\text{-C}^*\text{-isomorphism} S(f) \to S(g) \) is an \( S \) isomorphism (see Remark of Proposition 3.2.3). \( \blacksquare \)

**COROLLARY 2.2.3** Let
\[
A_0(T, E) := \{ \lambda \in \Lambda (T, E) \mid \lambda \text{ is a group homomorphism} \}.
\]
and for every $\lambda \in \Lambda_0(T, E)$ put

$$\varphi_\lambda : S(f) \to S(f), \quad X \mapsto U^*_\lambda X U_\lambda.$$ 

Then the map $\lambda \mapsto \varphi_\lambda$ is an injective group homomorphism.

By Proposition 1.1.4 c), $\Lambda_0(T, E)$ is the kernel of the map

$$\Lambda(T, E) \to \mathcal{F}(T, E), \quad \lambda \mapsto \delta \lambda,$$

so by Proposition 2.2.2, $\varphi_\lambda$ is well-defined. Thus only the injectivity of the map has to be proved. For $t \in T$ and $\zeta \in \mathcal{F}$, by Proposition 2.1.2 c),

$$U^*_\lambda V_t U_\lambda (\zeta \otimes e_1) = U^*_\lambda V_t (\zeta \otimes e_t) =
(\lambda(t) \otimes 1_K) V_t (\zeta \otimes e_1).$$

So if $\varphi_\lambda$ is the identity map then $\lambda(t) = 1_E$ for every $t \in T$.

**PROPOSITION 2.2.4** Let $F$ be a unital C*-algebra, $\varphi : E \to F$ a surjective C*-homomorphism, $g = \varphi \circ f \in \mathcal{F}(T, F)$, and $L = \bigoplus_{t \in T} \mathcal{F}$. We put for all $\xi \in H, \eta \in L$, and $X \in \mathcal{L}_E(H)$,

$$\tilde{\xi} = (\varphi \xi)_x \in L, \quad \tilde{X} \eta = \tilde{X} \xi \in L,$$

where $\zeta \in H$ with $\tilde{\xi} = \eta$ (Lemma 1.3.11 a),b) and Proposition 1.3.12 a)). Then

$$\tilde{X} = \sum_{t \in T} ((\varphi X_t \otimes 1_K) V^f_t \in S(g)$$

for every $X \in S(f)$ and the map

$$\tilde{\varphi} : S(f) \to S(g), \quad X \mapsto \tilde{X}$$

is a surjective C*-homomorphism, continuous with respect to the topologies $\Xi_k$, $k \in \{1, 2, 3\}$ such that

$$Ker \tilde{\varphi} = \{ X \in S(f) | t \in T \Rightarrow X_t \in Ker \varphi \}.$$

For $s, t \in T$ and $\xi \in H$,

$$((X_t \otimes 1_K) V^f_t \tilde{\xi})_s = (\varphi ((X_t \otimes 1_K) V^f_t \tilde{\xi}))_s = (\varphi X_t (\otimes 1_K) V^f_t \tilde{\xi})_s =
(\varphi X_t (\otimes 1_K) V^f_t \tilde{\xi})_s =
(\varphi X_t \otimes 1_K) V^f_t \tilde{\xi},$$

so by Lemma 1.3.11 b),

$$\tilde{(X_t \otimes 1_K) V^f_t} = ((\varphi X_t \otimes 1_K) V^f_t \tilde{\xi})_s.$$

By Theorem 2.1.9 b),

$$X = \sum_{t \in T} (X_t \otimes 1_K) V^f_t$$
so by the above and by Proposition 1.3.12 b),

\[ \bar{X} = \sum_{t \in T} ((\varphi X_\ell) \otimes 1_{K^\ell}) V^g_t \in S(g). \]

By Proposition 1.3.12 b), \( \bar{\varphi} \) is a surjective C**-homomorphism, continuous with respect to the topologies \( T_k (k \in \{1, 2, 3\}) \). The last assertion is easy to see.

COROLLARY 2.2.5  Let \( F \) be a unital C*-algebra, \( \varphi: E \to F \) a unital C*-homomorphism such that \( \varphi(Un E^\ell) \subseteq F^\ell \), \( g := \varphi \in \mathcal{F}(T, F) \), and \( L := \bigoplus_{t \in T} \hat{F} \). Then the map

\[ \bar{\varphi}: S_{\|f\|} \to S_{\|g\|}, \quad X \mapsto \sum_{t \in T} ((\varphi X_\ell) \otimes 1_{L^\ell}) V^g_t \]

is C*-homomorphism.

Put \( G := E/Ker \varphi \) and denote by \( \varphi_1: E \to G \) the quotient map and by \( \varphi_2: G \to F \) the corresponding injective C*-homomorphism.

By Proposition 2.2.4, the corresponding map

\[ \bar{\varphi}_1: S_{\|f\|} \to S_{\|\varphi_1 \circ f\|}, \quad X \mapsto \sum_{t \in T} ((\varphi X_\ell) \otimes 1_{L^\ell}) V^g_t \]

is a C*-homomorphism and by Theorem 2.1.9 k), the corresponding map

\[ \bar{\varphi}_2: S_{\|\varphi_1 \circ f\|} \to S_{\|g\|} \]

is also a C*-homomorphism. The assertion follows from \( \bar{\varphi} = \bar{\varphi}_2 \circ \bar{\varphi}_1 \).

PROPOSITION 2.2.6  Let \( T' \) be a group, \( K' := \hat{T}'(T') \), \( H' := \hat{E} \hat{\otimes} K' \), \( \psi: T \to T' \) a surjective group homomorphism such that

\[ \sup_{t' \in T'} \text{Card} \psi(t') \in \mathbb{N}, \]

and \( f' \in \mathcal{F}(T', E) \) such that \( f' \circ (\psi \times \psi) = f \). If we put

\[ X'_t := \sum_{t' \in \psi^{-1}(t')} X_{t'} \]

for every \( X \in S(f) \) and \( t' \in T' \) then the family \( ((X'_t \hat{\otimes} 1_{K'}) V^f_{t'})_{t' \in T'} \) is summable in \( L(E)(H')^\ell \) for every \( X \in S(f) \) and the map

\[ \bar{\psi}: S(f) \to S(f'), \quad X \mapsto X' := \sum_{t' \in T'} (X'_t \hat{\otimes} 1_{K'}) V^f_{t'} \]

is a surjective E-C**-homomorphism.

We may drop the hypothesis that \( \psi \) is surjective if we replace \( S \) by \( S_{\|\|} \).

Let \( X \in S(f) \). By Corollary 2.1.16 a), since \( \psi \) is surjective and

\[ \sup_{t' \in T'} \text{Card} \psi(t') \in \mathbb{N}, \]

it follows that the family \( ((X'_t \hat{\otimes} 1_{K'}) V^f_{t'})_{t' \in T'} \) is summable in \( L(E)(H')^\ell \) and therefore \( X' \in S(f') \).
Let \( X, Y \in \mathcal{S}(f) \). By Theorem 2.1.9 c), for \( t' \in T^* \),

\[
(X^*)_{t'} = \bar{f}'(t')(X_{t'^*})^* = \bar{f}'(t') \left( \sum_{t \in \Psi^{-1}(t'^*)} X_t \right)^* = \bar{f}'(t') \sum_{t \in \Psi^{-1}(t'^*)} (X_t)^* =
\]

\[
= \sum_{t \in \Phi^{-1}(t'^*)} \bar{f}(s)(X_{s'^*}) = \sum_{t \in \Phi^{-1}(t'^*)} (X^*)_t = (X^*)_{t'},
\]

\[
(XY)_{t'} = \bar{f}'(s, s'^*)X_{s'^*}Y_{s'^*} = \sum_{t \in \Phi^{-1}(t'^*)} \bar{f}(s, s'^*)X_{s'^*}Y_{s'^*} =
\]

\[
= \sum_{t \in \Phi^{-1}(t'^*)} \sum_{s \in \Phi^{-1}(s'^*)} \bar{f}(s, s'^*)X_{s'^*}Y_{s'^*} = \sum_{t \in \Phi^{-1}(t'^*)} (XY)_t = (XY)_{t'}.
\]

Thus \( \psi \) is a C*-homomorphism. The other assertions are easy to see.

The last assertion follows from Corollary 2.1.17 d).

\[\Box\]

**COROLLARY 2.2.7** If we use the notation of Proposition 2.2.6 and Corollary 2.2.5 and define \( \tilde{\psi} \) and \( \tilde{\psi} ' \) in an obvious way then \( \tilde{\psi} \circ \tilde{\psi} ' = \tilde{\psi} ' \circ \tilde{\psi} . \)

For \( X \in \mathcal{S}(f) \) and \( t' \in T^* \),

\[
(\tilde{\phi} \tilde{\psi} X)_{t'} = \tilde{\phi}(\tilde{\psi}X)_{t'} = \tilde{\phi} \sum_{t \in \Phi^{-1}(t')} X_t = \sum_{t \in \Phi^{-1}(t')} \tilde{\phi} X_t,
\]

\[
(\tilde{\psi} \tilde{\phi} X)_{t'} = \sum_{t \in \Phi^{-1}(t')} (\tilde{\phi} X)_{t'} = \sum_{t \in \Phi^{-1}(t')} \tilde{\psi} X_t,
\]

so

\[
\tilde{\psi} \circ \tilde{\psi} ' = \tilde{\psi} ' \circ \tilde{\psi} .
\]

\[\Box\]

**PROPOSITION 2.2.8** Let \( F \) be a unital C*-subalgebra of \( E \) such that \( f(s, t) \in F \) for all \( s, t \in T \). We denote by \( \psi : F \rightarrow E \) the inclusion map and put

\[
f^F : T \times T \rightarrow \text{Un} F, \quad (s, t) \mapsto f(s, t),
\]

\[
H^F := \bigoplus_{t \in T} E \otimes K, \quad \tilde{\psi} : H^F \rightarrow H, \quad \xi \mapsto (\psi \xi)_{t \in T}.
\]

Moreover we denote for all \( s, t \in T \) by \( u^s_t, \tilde{V}^s_t \) and \( \varphi^s_t \) the corresponding operators associated with \( F \) \( (f^s \in \mathcal{F}(T, F)) \). Let \( X \in \mathcal{S}_C(f) \) such that \( X(\tilde{\psi} \xi) \in \tilde{\psi}(H^F) \) for every \( \xi \in H^F \) and put

\[
X^F : H^F \rightarrow H^F, \quad \xi \mapsto \tilde{\xi}.
\]

where \( \tilde{\xi} \in H^F \) with \( \tilde{\xi}^F = X(\tilde{\psi} \xi) \), and \( X^F := (u^s_t)^* X^s_t u^s_t \in F \) (by the canonical identification of \( F \) with \( L_F(F) \)) for every \( t \in T \).

a) \( \tilde{\xi}, \eta \in H^F \Rightarrow (\tilde{\psi} \xi \tilde{\psi} \eta) = \psi(\xi \eta) \).

b) \( \tilde{\psi} \) is linear and continuous with \( \| \tilde{\psi} \| = 1 \).

c) \( X^F \) is linear and continuous with \( \| X^F \| = \| X \| \).

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d) For \( s, t \in T \),
\[
\psi \phi_{st}^F X^F = \phi_{st} X^F, \quad \phi_{st} X^F = \phi_{st} X^F \quad \text{and} \quad \psi \phi_{st}^F X^F = f^F (st^{-1}, t) X^F_{st^{-1}}.
\]

By a) and Proposition 2.1.6 b),
\[
\phi_{st} X^F = (X^F (1_F \otimes e_t)) (1_F \otimes e_t),
\]
and by Proposition 2.1.8,
\[
\phi_{st}^F X^F = \psi (X^F (1_F \otimes e_t)) (1_F \otimes e_t) = (\psi (X^F (1_F \otimes e_t))) (\psi (1_F \otimes e_t)) = (X^F (1_F \otimes e_t)) (1_F \otimes e_t) = \phi_{st} X^F.
\]

In particular
\[
\phi_{s} X^F = \psi \phi_{s}^F X^F = \phi_{s} X = X_t
\]

and by Proposition 2.1.8,
\[
\phi_{s} X^F = \phi_{s} X = f(st^{-1}, t) X_{st^{-1}} = \psi f(st^{-1}, t) X_{st^{-1}},
\]
\[
\phi_{s}^F X^F = f(st^{-1}, t) X_{st^{-1}}.
\]

e) By c) and Proposition 2.1.3 d), for \( \xi \in H^F \),
\[
\sum_{t \in T} u_t^F (u_t^F)^* \xi = \xi,
\]
\[
X^F \xi = X^F \sum_{t \in T} u_t^F (u_t^F)^* \xi = \sum_{t \in T} X^F u_t^F (u_t^F)^* \xi,
\]
\[
X^F \xi = \sum_{s \in T} \sum_{t \in T} u_s^F (u_t^F)^* X^F \xi = \sum_{s \in T} \sum_{t \in T} u_s^F ((u_t^F)^* X^F u_t^F) (u_t^F)^* \xi.
\]

By d) and Proposition 2.1.4 b),
\[
X^F \xi = \sum_{s \in T} \sum_{t \in T} u_s^F (u_t^F)^* f(st^{-1}, t) X^F_{st^{-1}} (u_t^F)^* \xi = \sum_{s \in T} \sum_{t \in T} u_s^F (u_t^F)^* V_{st^{-1}}^F \xi = \sum_{s \in T} \sum_{t \in T} u_s^F (X^F_{st^{-1}} (u_t^F)^* V_{st^{-1}}^F \xi = \sum_{s \in T} \sum_{t \in T} u_s^F (X^F_{st^{-1}} (u_t^F)^* V_{st^{-1}}^F \xi = \sum_{s \in T} \sum_{t \in T} u_s^F (X^F_{st^{-1}} (u_t^F)^*) (X^F_{st^{-1}} (u_t^F)^*) V_{st^{-1}}^F \xi = \sum_{s \in T} \sum_{t \in T} u_s^F (X^F_{st^{-1} \otimes 1_K}) V_{st^{-1}}^F \xi = \sum_{s \in T} \sum_{t \in T} u_s^F (X^F_{st^{-1} \otimes 1_K}) V_{st^{-1}}^F \xi.
\]

by Proposition 2.1.3 d), again. Thus
\[
X^F = \sum_{s \in T} \sum_{s \in T} (X^F_{st^{-1} \otimes 1_K}) V_{st^{-1}}^F \in S_C (f^F).
\]

f) For \( s, t \in T \), by d),
\[
(\psi ((X^F_{st^{-1} \otimes 1_K}) V_{st^{-1}}^F \xi))_s = (\psi (X^F (1_F \otimes e_t)) V^F_t \xi)_s = \psi (f(t, t^{-1} s) X^F_{st^{-1} \otimes 1_K}) = f(t, t^{-1} s) X_t (\psi (1_F \otimes e_t) ) = (X_t \otimes 1_K) V_t \psi \xi_s,
\]
\[
\psi ((X^F_{st^{-1} \otimes 1_K}) V_{st^{-1}}^F \xi) = (X_t \otimes 1_K) V_t \psi \xi.
\]
so by b) and e),

\[ X(\tilde{\psi}\xi) = \tilde{\varphi}(X^f\xi) = \tilde{\varphi} \left( \sum_{t \in T} (X_t^f \otimes 1_K) V_t^f \xi \right) = \sum_{t \in T} \tilde{\psi}((X_t^f \otimes 1_K) V_t^f \xi) = \sum_{t \in T} (X_t \otimes 1_K) V_t(\tilde{\psi}\xi). \]

\[ \square \]

**Proposition 2.2.9** Let \( F \) be a \( W^\ast \)-algebra such that \( E \) is a unital \( C^\ast \)-subalgebra of \( F \) generating it as \( W^\ast \)-algebra, \( \varphi : E \to F \) the inclusion map, and \( \xi := (\varphi \xi)_{t \in T} \in L \) for every \( \xi \in H \), where

\[ L := \bigotimes_{t \in T} F \cong F \otimes K. \]

a) \( \varphi(\text{Un } F') \subset \text{Un } F' \) and \( g = \varphi \text{ of } E \in F(T, F) \).

b) If

\[ \psi : \mathcal{L}_E(H) \to \mathcal{L}_F(L), \quad X \mapsto \tilde{X} \]

is the injective \( C^\ast \)-homomorphism defined in Proposition 1.3.9 b), then \( \psi(S_C(f)) \subset S_W(g) \), \( \psi(S_C(f)) \) generates \( S_W(g) \) as \( W^\ast \)-algebra, and for every \( X \in S_C(f) \) and \( t \in T \) we have \( (\tilde{X})_t = \varphi X_t \).

c) The following are equivalent for every \( Y \in S_W(g) \):

\begin{enumerate}
  
  c\textsubscript{1} \, Y \in \psi(S_C(f)).
  
  c\textsubscript{2} \, \xi \in H \Rightarrow Y \tilde{\xi} \in H.

  \text{If these conditions are fulfilled then}
  
  c\textsubscript{3} \, (Y \tilde{\xi})_t \in t \in H.
  
  c\textsubscript{4} \, (Y^*_t)_{t \in T} \in \mathcal{E}.
  
  c\textsubscript{5} \, \xi \in H \Rightarrow Y \tilde{\xi} = \sum_{t \in T} (Y_t \otimes 1_K) V_t^\beta \tilde{\xi} \in H.
\end{enumerate}

a) follows from the density of \( \varphi(E) \) in \( F \) (Lemma 1.3.8 a \Rightarrow c).

b) For \( x \in E, t \in T, \) and \( \xi \in H \),

\[ (((\varphi x) \otimes 1_K) V_t^f \tilde{\xi})_t = g(t, t^{-1}s)(\varphi x) \bar{\xi}_{t^{-1}s} = \varphi f(t, t^{-1}s) x \bar{\xi}_{t^{-1}s} = \varphi((x \otimes 1_K) V_t \xi) \]

so

\[ (((\varphi x) \otimes 1_K) V_t^f \tilde{\xi})_t = (x \otimes 1_K) V_t^f. \]

Let now \( X \in S(f) \). By Theorem 2.1.9 b),

\[ X = \sum_{t \in T} (X_t \otimes 1_K) V_t^f \]

so by the above and by Proposition 1.3.9 c) (and Theorem 2.1.9 d)),

\[ \tilde{X} = \sum_{t \in T} (X_t \otimes 1_K) V_t^f = \sum_{t \in T} (((\varphi X_t) \otimes 1_K) V_t^f)_{t \in S_W(g)} \]

so \( \psi(S_C(f)) \subset S_W(f) \). By Theorem 2.1.9 a), \( (\tilde{X})_t = \varphi X_t \) for every \( t \in T \).
Since $\varphi (E)$ is dense in $F_F$ (Lemma 1.3.8 a $\Rightarrow$ c) it follows that

$$R(g) \subset \varphi (R(f))$$

so $\psi (S(f))$ is dense in $S(g)$ and therefore generates $S(g)$ as W*-algebra (Lemma 1.3.8 c $\Rightarrow$ d).

$c_1 \Rightarrow c_2$ follows from the definition of $\psi$.

$c_2 \Rightarrow c_1$ follows from Proposition 2.2.8 e).

$c_2 \Rightarrow c_3 & c_4$ follows from Proposition 2.1.23 b).

$c_2 \Rightarrow c_5$ follows from Proposition 2.2.8 f).

\[ \blacksquare \]

**LEMMA 2.2.10** Let $E, F$ be W*-algebras, $G := E \hat{\otimes} F$, and

$$L := \bigoplus_{t \in T} G \cong \hat{G} \otimes K$$

a) If $z \in G^e$ then $z \otimes_1 1_K$ belongs to the closure of

$$\{ w \otimes_1 1_K | w \in E \hat{\otimes} F, \|w\| \leq 1 \}$$

in $L_c(L)_L$.

b) For every $y \in F$, the map

$$E_E^c \rightarrow G^e, \quad x \mapsto x \otimes y$$

is continuous.

a) By [1] Corollary 6.3.8.7, there is a filter $\mathcal{F}$ on $\{ w \in E \hat{\otimes} F | \|w\| \leq 1 \}$ converging to $z$ in $G^e$. By Lemma 1.3.2 b), for $(a, \xi, \eta) \in \hat{G} \times L \times L$,

$$\langle z \otimes_1 1_K, (a, \xi, \eta) \rangle = \lim_{w \rightarrow \mathcal{F}} \left( w, \sum_{t \in T} \xi_t a_t^* \right) = \lim_{w \rightarrow \mathcal{F}} \left( w \otimes_1 1_K, (a, \xi, \eta) \right)$$

which proves the assertion.

b) Let $(a_t, b_t)_{t \in I}$ be a finite family in $E \times F$. For $x \in E$,

$$\langle x \otimes y, \sum_{i \in I} a_i \otimes b_i \rangle = \sum_{i \in I} \langle x, a_i \rangle \langle y, b_i \rangle = \langle x, \sum_{i \in I} \langle y, b_i \rangle a_i \rangle.$$

Since $\{ x \otimes y / x \in E^e \}$ is a bounded set of $G$, the above identity proves the continuity.

\[ \blacksquare \]

**PROPOSITION 2.2.11** Let $F$ be a unital C**-algebra, $S$ a group, and $g \in F(S, F)$. We denote by $\otimes_s$ the spatial tensor product and put

$$G := E \otimes_s F \quad (\text{resp. } G := E \hat{\otimes} F),$$

$$L := \bigoplus_{(t, s) \in T \times S} G \cong \hat{G} \otimes T \cong (T \times S),$$

$$h : (T \times S) \times (T \times S) \rightarrow UnG^e, \quad ((t_1, s_1), (t_2, s_2)) \mapsto f(t_1, t_2) \otimes g(s_1, s_2).$$
a) \( h \in \mathcal{F}(T \times S, G) \), \( M = H \otimes L \).

\( L_\mathcal{E}(H) \otimes L_F(L) \subseteq L_\mathcal{E}(M) \) in the \( C^* \)-case,
\( L_\mathcal{E}(H) \otimes L_F(L) \approx L_\mathcal{E}(M) \) in the \( W^* \)-case.

b) For \( t \in T, s \in S, x \in E, y \in F \),

\[
((x \otimes 1_{P(T)}) V^t_1) \otimes ((y \otimes 1_{P(S)}) V^s_1) = ((x \otimes y) \otimes 1_{P(T \times S)}) V^{t \otimes s}_{(t,s)}.
\]

c) In the \( C^* \)-case, \( S_C(f) \otimes S_C(g) \approx S_C(h) \) and \( S_C(f) \otimes S_C(g) \approx S_C(h) \).
d) In the \( W^* \)-case, if \( z \in G^* \) and \( (t, s) \in T \times S \) then \( (z \otimes 1_{P(T \times S)}) V^h_{(t,s)} \) belongs to the closure of \( \{ (w \otimes 1_{P(T \times S)}) V^h_{(t,s)} | w \in (E \otimes F)^* \} \) in \( L_\mathcal{E}(M) \).
e) In the \( W^* \)-case, \( SW(f) \otimes SW(g) \approx SW(h) \).

a) \( h \in \mathcal{F}(T \times S, G) \) is obvious.

Let us treat the \( C^* \)-case first. For \( \xi, \xi' \in H \) and \( \eta, \eta' \in L \),

\[
(\xi, \eta') \otimes (\xi, \eta) = (\xi), (\xi' \otimes (\eta, \eta')) = \\
\sum_{(t, s) \in T \times S} (\xi, \eta' \otimes (\eta, \eta)) = \\
\sum_{(t, s) \in T \times S} (\xi' \otimes (\eta, \eta')) = \\
\sum_{(t, s) \in T \times S} (\xi \otimes \xi') \otimes (\eta, \eta'),
\]

so the linear map

\[
H \otimes L \rightarrow M, \quad \xi \otimes \eta \mapsto (\xi \otimes \eta)_{(t, s) \in T \times S}
\]

preserves the scalar products and it may be extended to a linear map \( \varphi : H \otimes L \rightarrow M \) preserving the scalar products.

Let \( z \in G, (t, s) \in T \times S \), and \( \varepsilon > 0 \). There is a finite family \( \{x_i, y_i\}_{i \in I} \) in \( E \times F \) such that

\[
\| \sum_{i \in I} x_i \otimes y_i - z \| < \varepsilon.
\]

Then

\[
\| \sum_{i \in I} (x_i \otimes e_1) \otimes (y_i \otimes e_1) - z \otimes e_{(t, s)} \| < \varepsilon.
\]

so \( z \otimes e_{(t, s)} \in \overline{\varphi[H \otimes L]} = \varphi(H \otimes L) \). It follows that \( \varphi \) is surjective and so \( H \otimes L \approx M \).

The proof for the inclusion \( L_\mathcal{E}(H) \otimes_M L_F(L) \subseteq L_\mathcal{E}(M) \) can be found in [6] page 37.

Let us now discuss the \( W^* \)-case. \( \mathcal{F} \otimes \tilde{F} \approx \mathcal{G} \) follows from [2] Proposition 1.3 e), \( M = H \otimes L \) follows from [3] Corollary 2.2, and \( L_\mathcal{E}(H) \otimes L_F(L) \approx L_\mathcal{E}(M) \) follows from [2] Theorem 2.4 d) or [3] Theorem 2.4.

b) For \( t_1, t_2 \in T, s_1, s_2 \in S, \xi \in \mathcal{E}, \) and \( \eta \in \mathcal{F} \), by Proposition 2.1.2 f) and [3] Corollary 2.11,

\[
(((x \otimes 1_{P(T)}) V^t_1) \otimes ((y \otimes 1_{P(S)}) V^s_1)) \otimes (\xi \otimes \eta) V^h_{(t_1, s_1)} = \\
= ((x \otimes 1_{P(T)}) V^t_1) \otimes (\xi) V^h_{(t_1, s_1)} = \\
= ((y \otimes 1_{P(S)}) V^s_1) \otimes (\eta) V^h_{(t_1, s_1)} = \\
= ((x \otimes y) (\otimes 1_{P(T \times S)}) V^h_{(t_1, s_1)}) \otimes (\xi) (\otimes 1_{P(S)}) V^h_{(t_2, s_2)} = \\
= ((x \otimes y) (\otimes 1_{P(T \times S)}) V^h_{(t_1, s_1)}) \otimes (\eta) (\otimes 1_{P(S)}) V^h_{(t_2, s_2)} = \\
= ((x \otimes y) (\otimes 1_{P(T \times S)}) V^h_{(t_1, s_1)}) \otimes ((y \otimes 1_{P(S)}) V^h_{(t_2, s_2)}) (\eta) (\otimes 1_{P(S)}) V^h_{(t_2, s_2)}. \]
We put
\[ u = (x \otimes 1_{F(T)}) V^I \otimes ((y \otimes 1_{F(S)}) V^I) - (z \otimes 1_{F(T \times S)}) V^I \in \mathcal{L}_{c}(M). \]

By the above, \( u(\zeta \odot e_r) = 0 \) for all \( \zeta \in \hat{\mathcal{E}} \odot \hat{F} \) and \( r \in T \times S \).

Let us consider the CG-case first. Since \( \hat{\mathcal{E}} \odot \hat{F} \) is dense in \( \hat{G} \), we get \( u(\zeta \odot e_r) = 0 \) for all \( \zeta \in G \) and \( r \in T \times S \). For \( \zeta \in M \), by [1] Proposition 5.6.4.1 e),
\[ u_{\zeta} = u \left( \sum_{r \in T \times S} (\zeta \odot e_r) \right) = \sum_{r \in T \times S} u(\zeta \odot e_r) = 0, \]
which proves the assertion in this case.

Let us consider now the W*-case. Let \( z \in \mathcal{G}^\# \) and \( r \in T \times S \) and let \( \mathcal{R} \) be a filter on \( (E \odot F)^\# \) converging to \( z \) in \( \mathcal{G} \) ([1] Corollary 6.3.8.7). For \( \eta \in M, a \in \hat{\mathcal{E}}, \) and \( r \in T \times S \),
\[ \langle z \odot e_r, (a, \tilde{\eta}) \rangle = \langle (z \odot e_r) \eta, a \rangle = \langle \eta^*, z, a \rangle = \langle z, a \eta^* \rangle = \lim_{w \in \mathcal{R}} \langle w \odot e_r, (a, \eta) \rangle = \lim_{w \in \mathcal{R}} w \odot e_r = z \odot e_r \]
in \( M_{\mathcal{M}} \). Since \( u : M_{\mathcal{M}} \to M_{\mathcal{M}} \) is continuous ([1] Proposition 5.6.3.4 c)), we get by the above \( u(\zeta \odot e_r) = 0 \). For \( \zeta \in M \) it follows by [1] Proposition 5.6.4.6 c),
\[ u_{\zeta} = u \left( \sum_{r \in T \times S} (\zeta \odot e_r) \right) = \sum_{r \in T \times S} u(\zeta \odot e_r) = 0 \]
which proves the assertion in the W*-case.

c) By b), \( \mathcal{R}(f) \odot \mathcal{R}(g) \subset \mathcal{R}(h) \) so by a),
\[ S_\|f\| \odot S_\|g\| \subset S_{\|h\|}, \quad S_C(f) \odot S_C(g) \subset S_C(h), \]
\[ S_\|f\| \odot S_\|g\| \subset S_{\|h\|}, \quad S_C(f) \odot S_C(g) \subset S_C(h). \]

Let \( z \in \mathcal{G}^\#, (t, s) \in T \times S, \) and \( \varepsilon > 0 \). There is a finite family \( (x_i, y_i)_{i \in I} \) in \( E \times F \) such that
\[ \left\| \sum_{i \in I} (x_i \otimes y_i) \right\| < 1, \quad \left\| \sum_{i \in I} (x_i \otimes y_i) - z \right\| < \varepsilon. \]

By b),
\[ \left\| \sum_{i \in I} ((x_i \otimes 1_{F(T)}) V^I) \otimes ((y_i \otimes 1_{F(S)}) V^I) - (z \otimes 1_{F(T \times S)}) V^I \right\| < \varepsilon \]
and so by a),
\[ \mathcal{R}(h) \subset \mathcal{R}(f) \odot \mathcal{R}(g) \subset \mathcal{R}(f) \odot \mathcal{R}(g), \]
\[ S_{\|h\|} \subset S_{\|f\|} \odot S_{\|g\|}, \quad S_C(h) \subset S_C(f) \odot S_C(g). \]

d) By a) and Lemma 2.2.10 a), there is a filter \( \mathcal{R} \) on
\[ \{ w \otimes 1_{F(T \times S)} | w \in (E \odot F)^\# \} \]
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converging to $z \otimes 1_{P(T \times S)}$ in $L(G)_M$. For $z, \eta \in M$ and $a \in \mathbb{G}$,

$$(z \otimes 1_{P(T \times S)})_{(x, \eta)} = (z \otimes 1_{P(T \times S)})_{(x, \eta)} = \lim_{w \uparrow H} (w \otimes 1_{P(T \times S)})_{(x, \eta)},$$

which proves the assertion.

e) By Theorem 2.1.9 h),

$$\left( \frac{R(f)}{R(g)} \right)^\# = S_W(f)^\# \subset L_E(H), \quad \left( \frac{L}{R(g)} \right)^\# = S_W(g)^\# \subset L_L(L).$$

By b), $\mathcal{R}(f) \cap \mathcal{R}(g) \subset \mathcal{R}(h)$, so by Lemma 2.2.10 b),

$$S_W(f)^\# \cap S_W(g)^\# \subset S_W(h)^\#.$$

By [3] Proposition 2.5,

$$S_W(f) \otimes S_W(g) \approx S_W(f) \otimes S_W(g) \subset S_W(h).$$

For $x, y \in E, f \in F$, and $(t, s) \in T \times S$, by b),

$$((x \otimes y) \otimes 1_{P(T \times S)})_{(x, \eta)} = ((x \otimes 1_{P(T)}) \otimes (y \otimes 1_{P(S)}) \otimes 1)_{(x, \eta)} \in S_W(f) \otimes S_W(g).$$

Let $z \in G^\#$. By d), there is a filter $\mathcal{R}$ on

$$\{ (w \otimes 1_{P(T \times S)})_{(x, \eta)} | w \in (E \otimes F)^\# \}$$

converging to $(z \otimes 1_{P(T \times S)})_{(x, \eta)}$ in $L(G)_M$, so by the above

$$(z \otimes 1_{P(T \times S)})_{(x, \eta)} \in S_W(f) \otimes S_W(g).$$

We get

$$\mathcal{R}(h) \subset S_W(f) \otimes S_W(g), \quad S_W(h) \subset S_W(f) \otimes S_W(g),$$

$$S_W(h) = S_W(f) \otimes S_W(g).$$

**COROLLARY 2.2.12** Let $n \in \mathbb{N}$ and $g : T \times T \rightarrow Un(E_{n,n})^\#_c, \quad (s, t) \mapsto \hat{\delta}_{ij}(s, t)|_{i, j \in \mathbb{N}}, n$.

a) $(S(f))_{n,n} \approx S(g)$, \quad $(S(f))_{n,n} \approx S(g)$.

b) Let us denote by $\rho : S(g) \rightarrow (S(f))_{n,n}$ the isomorphism of a). For $X \in S(g)$, $t \in T$, and $i, j \in \mathbb{N}$,

$$(\rho X)_{ij} = (X_{ij}).$$

a) Take $F := \mathbb{K}_{n,n}$ and $S := \{1\}$ in Proposition 2.2.11. Then $G = E_{n,n}$ and $g : T \times T \rightarrow Un(G)^\#_c, \quad (s, t) \mapsto f(s, t) \otimes 1_F.$

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By Proposition 2.2.11 c),
\[ S(g) = S(f) \otimes K_{n,n} \approx (S(f))_{n,n}, \]
\[ S_{\|\|_{\|\|}}(g) = S_{\|\|_{\|\|}}(f) \otimes K_{n,n} \approx (S_{\|\|_{\|\|}}(f))_{n,n}. \]

b) By Theorem 2.1.9 b),
\[ X = \sum_{s=t}^{2} (X_{s} \otimes 1_{E})V_{s}^{q} \]
so
\[ (pX)_{ij} = \sum_{s=t}^{2} ((X_{s})_{ij} \otimes 1_{E})V_{s}^{q}, \]
\[ ((pX)_{ij})_{t} = (X_{t})_{ij} \]
by Theorem 2.1.9 a)

COROLLARY 2.2.13 Let \( n \in \mathbb{N} \). If \( \mathbb{K} = \mathbb{C} \) (resp. if \( n = 4^{n} \) for some \( m \in \mathbb{N} \)) then there is an \( f \in \mathcal{F}(\mathbb{Z} \times n, E) \) (resp. \( f \in \mathcal{F}(\mathbb{Z} \times n, E) \)) such that
\[ R(f) = S(f) \approx E_{n,n}. \]

By [1] Proposition 7.1.4.9 b), d) (resp. [1] Theorem 7.2.2.7 i), k)) there is a \( g \in \mathcal{F}(\mathbb{Z} \times n, C) \) (resp. \( g \in \mathcal{F}(\mathbb{Z} \times n, C) \)) such that
\[ S(g) \approx C_{n,n} \quad (\text{resp. } S(g) \approx K_{n,n}). \]

If we put
\[ f : (\mathbb{Z} \times n, n) \times (\mathbb{Z} \times n, n) \rightarrow \mathcal{U}nE^{*}, \quad (s, t)\mapsto g(s, t) \otimes 1_{E} \]
\[ (\text{resp. } f : (\mathbb{Z} \times n, n) \rightarrow \mathcal{U}nE^{*}, \quad (s, t)\mapsto g(s, t) \otimes 1_{E}) \]
then by Proposition 2.2.11 a), e), \( f \in \mathcal{F}(\mathbb{Z} \times n, E) \) (resp. \( f \in \mathcal{F}(\mathbb{Z} \times n, E) \)) and
\[ S(f) = S(g) \otimes E \approx K_{n,n} \otimes E \approx E_{n,n}. \]

COROLLARY 2.2.14 Let \( F \) be a unital \( C^{*} \)-algebra, \( G = E \otimes F \), and
\[ h : T \times T \rightarrow \mathcal{U}nG^{*}, \quad (s, t)\mapsto f(s, t) \otimes 1_{F}. \]
Then \( h \in \mathcal{F}(T, G) \) and
\[ S_{\|\|_{\|\|}}(h) = S_{\|\|_{\|\|}}(f) \otimes F, \quad S(h) = S(f) \otimes F. \]

COROLLARY 2.2.15 If \( E \) is a \( W^{*} \)-algebra then the following are equivalent:

a) \( E \) is semifinite.

b) \( S_{W}(f) \) is semifinite.

a \Rightarrow b. Assume first that there are a finite \( W^{*} \)-algebra \( F \) and a Hilbert space \( L \) such that \( E \approx F \otimes \mathcal{L}(L) \). Put
\[ g : T \times T \rightarrow \mathcal{U}nF^{*}, \quad (s, t)\mapsto f(s, t). \]

By Corollary 2.2.14,
\[ S_{W}(f) \approx S_{W}(g) \otimes \mathcal{L}(L). \]

By Corollary 2.1.11 c), \( S_{W}(g) \) is finite and so \( S_{W}(f) \) is semifinite.
The general case follows from the fact that \( E \) is the \( C^{*} \)-direct product of \( W^{*} \)-algebras of the above form ([8] Proposition V.1.40).

b \Rightarrow a. \( E \) is isomorphic to a \( W^{*} \)-subalgebra of \( S_{W}(f) \) (Theorem 2.1.9 h)) and the assertion follows from [8] Theorem V.2.15. ■
**PROPOSITION 2.2.16** Let $S$, $T$ be finite groups and $g \in \mathcal{F}(S, S(f))$ and put $L := \tilde{f}(S)$, $M := \tilde{f}(S \times T)$, and

$$ h : (S \times T) \times (S \times T) \rightarrow \text{Uns}(f)' $$

Then $h \in \mathcal{F}(S \times T, S(f))$ and the map

$$ \varphi : S(g) \rightarrow S(h), \quad X \mapsto \sum_{(x) \in S(S(g))} ((X)_t \otimes 1_u)V_{(x,t)}^h $$

is an $S(f)$-C*-isomorphism.

For $X, Y \in S(g)$, $Z \in S(f)$, and $(s, t) \in S \times T$, by Theorem 2.1.9 c),g),

$$ (\varphi(X^*))_{(x,t)} = ((X^*)_t) = ((g(s)(X_{r^{-1}}))^*)_t = ((g(s))^*_t(X_{r^{-1}}))^* = \tilde{f}(t)((g(s))(X_{r^{-1}}))^* = \tilde{h}(s, t)((\varphi X)^*_{(s,t)})) = ((\varphi X^*)_{(x,t)}), $$

$$ (\sum_{(r, u) \in S \times T} g(r, r^{-1}s)f(u, u^{-1}t)(X)_r(Y_{r^{-1}u})_{u^{-1}t} = \sum_{r \in S} g(r, r^{-1}s)(X)_r(Y_{r^{-1}u})_{u^{-1}t} = \sum_{r \in S} g(r, r^{-1}s)X_rY_{r^{-1}u} = \sum_{r \in S} g(r, r^{-1}s)X_rY_{r^{-1}u} $$

so

$$ \varphi(X^*) = (\varphi X)^*, \quad \varphi(XY) = (\varphi X)(\varphi Y), \quad \varphi(ZX) = Z\varphi(X) $$

and $\varphi$ is an $S(f)$-C*-homomorphism.

If $X \in S(g)$ with $\varphi X = 0$ then for $(s, t) \in S \times T$,

$$ (X)_t = (\varphi X)_{(x,t)} = 0, \quad \varphi = 0, \quad X = 0 $$

so $\varphi$ is injective.

Let $x \in E$ and $(s, t) \in S \times T$. Put

$$ Z := (x \otimes 1_u)V_{(x,t)}^h \in S(f), \quad X := (Z \otimes 1_u)V_{(x,t)}^h \in S(g). $$

Then for $(r, u) \in S \times T$,

$$ (\varphi X)_{(r, u)} = (X)_u = \delta_{r, s}\delta_{u, t} $$

so

$$ \varphi X = (x \otimes 1_u)V_{(x,t)}^h $$

and $\varphi$ is surjective. 

- **PROPOSITION 2.2.17** Let $S$ be a finite subgroup of $T$ and $g := f(S \times S)$. We identify $S(g)$ with the $E$-C**-subalgebra \( \{ ZeS(f) | t \in T, S = \emptyset \} \) of $S(f)$ (Corollary 2.1.17 e)), Let $X \in S(f) \cap S(g)^{c}$, $P_+ := XX'$, and $P_- := XX'$ and assume $P_+ \in PrS(f)$. 

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a) $P_\pm \in S(g)^\gamma$.

b) The map

$$\varphi_\pm : S(g) \to P_\pm S(f) P_\pm, \quad Y \mapsto P_\pm Y P_\pm$$

is a unital $C^{**}$-homomorphism.

c) For every $Z \in \varphi_+(S(g))$, $XZ^* \in \varphi_-(S(g))$ and the map

$$\psi : \varphi_+(S(g)) \to \varphi_-(S(g)), \quad Z \mapsto XZ^*$$

is a $C^*$-isomorphism with inverse

$$\varphi_-(S(g)) \to \varphi_+(S(g)), \quad Z \mapsto X^* Z$$

such that $\varphi_- = \psi \circ \varphi_+$.

d) If $p \in PrS(g)$ then

$$(X(\varphi_-, p))^*(X(\varphi_-, p)) = \varphi_-, p, \quad (X((\varphi_-, p))(X(\varphi_-, p)))^* = \varphi_-, p.$$

e) If $\varphi_-$ is injective then $\varphi_+$ is also injective, the map

$$E \to P_\pm S(f) P_\pm, \quad x \mapsto P_\pm (x \otimes 1_k) P_\pm$$

is an injective unital $C^{**}$-homomorphism, $P_\pm S(f) P_\pm$ is an $E$-$C^{**}$-algebra, $\varphi_\pm(S(g))$ is an $E$-$C^{**}$-subalgebra of it, and $\varphi_+$ and $\psi$ are $E$-$C^{**}$-homomorphisms.

f) The above results still hold for an arbitrary subgroup $S$ of $T$ if we replace $S$ by $S_\parallel$.

a) follows from the hypothesis on $X$.

b) follows from a).

c) Let $Y \in S(g)$ with $Z = P_\pm Y P_\pm$. By the hypotheses of the Proposition,

$$XZ^* = XP_\pm YP_\pm X^* = XX^* YX^* X^* =$$

$$= XX^* Y^* X^* X = P_\pm P_\pm \epsilon \varphi_-(S(g))$$

and $\psi$ is a $C^*$-isomorphism. The other assertions follow from

$$X^*(XZ^*)X = P_\pm ZP_\pm = P_\pm YP_\pm.$$

d) By b) and c),

$$\begin{align*}
(X(\varphi_-, p))^*(X(\varphi_-, p)) &= (\varphi_-, p)X^* X(\varphi_-, p) = (\varphi_-, p)P_\pm (\varphi_-, p) = \varphi_-, p, \\
(X(\varphi_-, p))(X(\varphi_-, p))^* &= X(\varphi_-, p)(\varphi_-, p)^* X^* = X(\varphi_-, p)X^* = \psi_-, p = \varphi_-, p.
\end{align*}$$

e) follows from b), c), and Lemma 18.

f) follows from Corollary 2.1.17 d).

Remark. Even if $\varphi_\pm$ is injective $P_\pm S(f) P_\pm$ is not an $E$-$C^*$-subalgebra of $S(f)$.

**Theorem 2.2.18** Let $S$ be a finite subgroup of $T$, $L := \tilde{f}(S)$, $g \in f(S \times S)$, $\omega : \mathbb{Z} \times \mathbb{Z} \to T$ an injective group homomorphism such that $S \cap \omega(\mathbb{Z} \times \mathbb{Z}) = \{1\}$,

$$a = \omega(1, 0), \quad b = \omega(0, 1), \quad c = \omega(1, 1), \quad \alpha_1 = f(a, a), \quad \alpha_2 = f(b, b),$$

$\beta_1, \beta_2 \in \text{Un} E^*$ such that $\alpha_1 \beta_1^2 + \alpha_2 \beta_2^2 = 0$,

$$\gamma = \frac{1}{2}(\alpha_1^2 \beta_1^2 - \alpha_2^2 \beta_2^2) = \alpha_1^2 \beta_1^2 - \alpha_2^2 \beta_2^2, \quad$$

$$X = \frac{1}{2}((\beta_1 \otimes 1_k) V_T + (\beta_2 \otimes 1_k) V_B), \quad P_\pm := X^* X, \quad P_\mp := X X^*.$$
We assume \( f(s, c) = f(c, s) \) and \( cs = sc \) for every \( s \in S \), and \( f(a, b) = -f(b, a) = 1_E \). Moreover we consider \( S(g) \) as an \( E\text{-C}^*\)-subalgebra of \( S(f) \) (Corollary 2.1.17 e)).

a) We have
\[
\begin{align*}
f(a, c) &= -f(c, a) = -\alpha_1, & f(b, c) &= -f(c, b) = -\alpha_2, & f(c, c) &= -\alpha_1\alpha_2, \\
y^* = -\alpha_1^*\alpha_2^*, & y'^* \in S(g)^c.
\end{align*}
\]

b) We have
\[
P_\pm = \frac{1}{2}(V^f_1 \pm (\gamma_1^*1_\theta)V'^f_1) \in S(g)^c \cap \text{Pr} S(f), 
\]
\[
P_\pm = V^f_1, P_+ = V^f_1, P_-, = 0, 
\]
\[
X^2 = 0, XP_+ = X, P_+ X = X, P_+ X = XP_+ = 0, X + X^* \in \text{Un} S(f), 
\]
\[
Y \in S(g) \Rightarrow XYX = 0.
\]

c) The map
\[
E \to P_\pm S(f)P_\pm, \quad x \mapsto (x \otimes 1_\theta) P_\pm
\]
is a unital injective \( C^*\)-homomorphism; we shall consider \( P_\pm S(f)P_\pm \) as an \( E\text{-C}^*\)-algebra using this map.

d) The maps
\[
\varphi_+: S(g) \to P_+ S(f)P_+, \quad Y \mapsto P_+ Y P_+, 
\]
\[
\varphi_-: S(g) \to P_- S(f)P_-, \quad Y \mapsto (Y + \gamma_0) Z(X + X^*)
\]
are orthogonal injective \( E\text{-C}^*\)-homomorphisms and \( \varphi_+ + \varphi_- \) is an injective \( E\text{-C}^*\)-homomorphism. If \( Y_1, Y_2 \in \text{Un} S(g) \) (resp. \( Y_1, Y_2 \in \text{Pr} S(g) \)) then \( \varphi_+ Y_1 + \varphi_- Y_2 \in \text{Un} S(f) \) (resp. \( \varphi_+ Y_1 + \varphi_- Y_2 \in \text{Pr} S(f) \)). Moreover the map
\[
\psi: S(f) \to S(f), \quad \psi = (X + X^*) Z(X + X^*)
\]
is an \( E\text{-C}^*\)-isomorphism such that
\[
\psi^{-1} = \psi, \quad \psi(P_\pm S(f)P_\pm) = P_- S(f) P_-, \quad \psi \circ \varphi_+ = \varphi_-.
\]
If \( \mathbb{K} = \mathbb{C} \) then \( X + X^* \) is homotopic to \( V_1 \) in \( \text{Un} S(f) \) and \( \psi \) is homotopic to the identity map of \( S(f) \). Using this homotopy we find that \( \varphi_+ Y \) is homotopic in the above sense to \( \varphi_+ Y \) for every \( Y \in S(g) \) and \( \varphi_+ Y_1 + \varphi_- Y_2, \varphi_+ Y_1 + \varphi_- Y_2, \varphi_+ Y_1 + \varphi_- Y_2, \varphi_+ Y_1 + \varphi_- Y_2 \), and \( \varphi_+ (Y_2 Y_1 + P_-) \) are homotopic in the above sense for all \( Y_1, Y_2 \in S(g) \).

e) Let \( s \in S \) such that \( sa = as \). Then
\[
sb = bs, \quad f(sc, c) f(s, c) = -\alpha_1\alpha_2, 
\]
\[
f(sa, c) f(c, sa)^* = -1_E, \quad f(a, s) f(s, a)^* = f(b, s) f(s, b)^*.
\]

f) If \( sa = as \) for every \( s \in S \) then the map
\[
S \times (\mathbb{Z} \times \mathbb{Z}) \to T, \quad (s, r) \mapsto s(\omega r)
\]
is an injective group homomorphism.

g) If \( T \) is generated by \( S \cup \omega (\mathbb{Z} \times \mathbb{Z}) \) and \( sa = as \) for every \( s \in S \) then \( \varphi_+ \) and \( \varphi_- \) are \( E\text{-C}^*\)-isomorphisms with inverse
\[
P_\pm S(f)P_\pm \to S(g), \quad Z \to 2 \sum_{x \in S} (x \otimes 1_\theta) V^g_x,
\]

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where

$$\psi_- : S(g) \rightarrow P_- S(f) P_-, \quad Y \mapsto P_- Y P_-.$$  

h) If \( sa = as \) and \( f(s, a) \) for every \( s \in S \) then \( X \in S(g)^\vee \), \( \varphi \cdot Y = P \cdot Y \) for every \( Y \in S(g) \), and there is a unique \( S(g) \)-\( C^\ast \)-homomorphism \( \phi : S(g)^{\vee \times \vee} \rightarrow S(f) \) such that

$$\phi \left[ \begin{array}{ccc} 0 & 0 & 0 \\ \alpha_1 \beta_1^\ast & 1 & 0 \end{array} \right] = X.$$  

\( \phi \) is injective and

$$\phi \left[ V_1^g 0 0 \right] = P_+, \quad \phi \left[ 0 0 V_1^g \right] = P_-.$$  

i) If \( sa = as \) and \( f(a, s) \) for all \( s \in S \) and if \( T \) is generated by \( S \cup \omega(\mathbb{Z} \times \mathbb{Z}) \) then \( \phi \) is an \( S(g) \)-\( C^\ast \)-isomorphism and

\[
\phi^{-1} V_1^f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \phi^{-1} V_c^f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

and for every \( s \in S \)

$$\phi^{-1} V_1^f = \begin{bmatrix} V_1^g & 0 \\ 0 & 0 \end{bmatrix}, \quad \phi^{-1} V_c^f = \begin{bmatrix} 0 & V_1^g \\ V_1^c & 0 \end{bmatrix}.$$  

j) The above results still hold for an arbitrary subgroup \( S \) of \( T \) if we replace \( S \) with \( S_{\| \|} \).

a) By the equation of the Schur functions,

$$f(a, a) = f(a, c) f(a, b), \quad f(a, b) f(c, a) = f(a, c) f(b, a), \quad f(a, b) f(c, b) = f(b, b),$$

and so

$$\alpha_1 = f(a, c), \quad f(c, a) = -f(a, c) = -\alpha_1, \quad f(c, b) = \alpha_2,$$

$$-\alpha_2 = -f(c, b) = f(b, c), \quad f(c, c) = \alpha_1 f(b, c) = -\alpha_1 \alpha_2.$$  

For \( s \in S \), by Proposition 2.1.2 b),

$$V_1^f V_c^f = (f(s, c) \otimes 1_k) V_1^c = (f(s, c) \otimes 1_k) V_c^f = V_1^f V_c^f$$

and so \( V_c^f \in S(g)^\vee \) (by Proposition 2.1.2 d)).
b) By Proposition 2.1.2 b), d), e) (and Corollary 2.1.22 c)),

$$X' = \frac{1}{2}((\alpha_1^2 \beta_1^2) \otimes 1_k) V_a^f + ((\alpha_2^2 \beta_2^2) \otimes 1_k) V_b^f,$$

$$P_+ = \frac{1}{4} (2V_1^f + ((\alpha_1^2 \beta_1^2) \otimes 1_k) V_f^c - ((\alpha_2^2 \beta_2^2) \otimes 1_k) V_f^c) = \frac{1}{2}(V_1^f + (\gamma \otimes 1_k) V_f^c),$$

$$P_- = \frac{1}{4} (2V_1^f + ((\beta_1^2 \alpha_1^2) \otimes 1_k) V_f^c - ((\beta_2^2 \alpha_1^2) \otimes 1_k) V_f^c) = \frac{1}{2}(V_1^f - (\gamma \otimes 1_k) V_f^c).$$

By a),

$$P^\pm_+ = \frac{1}{2}(V_1^f + (\gamma \otimes 1_k)((-\alpha_1^2 \alpha_2^2) \otimes 1_k) V_f^c) = P_\pm,$$

$$P^\pm_- = \frac{1}{4}(V_1^f + 2(\gamma \otimes 1_k) V_f^c + (\gamma^2 \otimes 1_k)((-\alpha_1^2 \alpha_2^2) \otimes 1_k) V_f^c) =$$

$$= \frac{1}{2}(V_1^f + (\gamma \otimes 1_k) V_f^c) = P_\pm.$$

so, by a) again, $P_\pm \in S(g) \cap PrS(f)$. By Proposition 2.1.2 b), d),

$$X^2 = \frac{1}{4}((\beta_1^2 \alpha_1^2 + \beta_2^2 \alpha_2^2) \otimes 1_k) V_1^f + ((\beta_1 \beta_2) \otimes 1_k)(V_a^f V_b^f + V_b^f V_a^f) = 0,$$

$$(X + X')^2 = XX' + X'X + X^2 = P_+ + P_- = V_1^f.$$

For the last relation we remark that by the above,

$$XYX = X(P_+ + P_-)YX = XP_+ YX = XYP_+ X = 0.$$ 

c) follows from b) and Lemma 1.3.2.

d) By b) and c), the map $\varphi_\pm$ is an $E$-$C^\ast$-homomorphism. Let $Y \in S(g)$ with $\varphi_\pm Y = 0$. By b), $Y = \pm Y(\gamma \otimes 1_k) V_f^c$ so by Proposition 2.1.2 b), d) and Theorem 2.1.9 b),

$$\sum_{s \in S} (Y_s \otimes 1_k) V_s^f = \mp Y(\gamma \otimes 1_k) V_f^c = \mp \sum_{s \in S} ((Y_s \gamma f(s, c) \otimes 1_k) V_s^f,$$

which implies $Y_s = 0$ for every $s \in S$ (Theorem 2.1.9 a)). Thus $\varphi_\pm$ is injective. It follows that $\varphi_+ + \varphi_-$ is also injective.

Assume first $Y_1, Y_2 \in UnS(g)$. By b),

$$(\varphi_+ Y_1 + \varphi_- Y_2)(\varphi_+ Y_1 + \varphi_- Y_2) = (\varphi_+ Y_1^2 + \varphi_- Y_2^2)(\varphi_+ Y_1 + \varphi_- Y_2) =$$

$$= \varphi_+ (Y_1^2 Y_1) + \varphi_- (Y_2^2 Y_2) = P_+ + P_- = V_1^f.$$

Similarly $(\varphi_+ Y_1 + \varphi_- Y_2)(\varphi_+ Y_1 + \varphi_- Y_2) = V_1^f$. The case $Y_1, Y_2 \in PrS(g)$ is easy to see.

By b), $\psi$ is an $E$-$C^\ast$-isomorphism with

$$\psi^{-1} = \psi, \quad \psi P_+ = (X + X')X(X + X') = XX'XX' = P_-.$$

Moreover for $Y \in S(g),$

$$\psi \varphi_+ Y = (X + X')P_+ YP_+(X + X') = XYX = \varphi_- Y.$$

Assume now $\mathbb{K} = \mathbb{C}$. By b), $X + X' \in UnS(f)$. Being selfadjoint its spectrum is contained in $\{-1, +1\}$ and so it is homotopic to $V_1^f$ in $UnS(f)$. 

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Theorem 2.1.9 b) (and Corollary 1.3.7 d)),

From $\omega S f)$ Since by Proposition 2.1.2 b),

and so by Theorem 2.1.9 a),

\[
f(s, c) f(sc, c) = f(s, 1) f(c, c) = -\alpha_1 \alpha_2,
\]

\[
f(s, a) f(sa, c) = f(s, b) f(a, c) = \alpha f(s, b),
\]

\[
f(c, as) f(a, s) = f(c, a) f(b, s) = -\alpha_1 f(b, s),
\]

\[
f(c, bs) f(b, s) = f(c, b) f(a, s) = \alpha f(a, s),
\]

\[
f(s, c) f(sc, b) = f(s, a) f(c, b) = \alpha f(s, a),
\]

\[
f(c, s) f(cs, b) = f(c, sb) f(s, b)
\]

so

\[
f(sa, c) f(c, as)^* = -f(s, b) f(s, a)^* f(b, s)^* f(a, s) =
\]

\[
= -f(s, c) f(cs, b) f(c, sb)^* \alpha_2 f(s, c)^* f(sc, b)^* \alpha_2 f(c, bs) = -1_E.
\]

From

\[
f(s, c) f(sc, a) = f(s, b) f(c, a), \quad f(c, a) f(b, s) = f(c, as) f(a, s),
\]

\[
f(c, s) f(cs, a) = f(c, sa) f(s, a)
\]

we get

\[
f(a, s) f(s, a)^* = f(b, s) f(s, b)^*.
\]

f) Since $S$ and $\omega(Z_2 \times Z_2)$ commute, the map is a group homomorphism. If $s_{oor} = 1$ for $(s, r) \in S \times (Z_2 \times Z_2)$ then $s_{oor} = s^{-1} \in S \cap \omega(Z_2 \times Z_2)$, which implies $s = 1$ and $r = (0, 0)$. Thus this group homomorphism is injective.

g) By e) and the hypothesis of f), for every $t \in T$ there are uniquely $s \in S$ and $d \in \{1, a, b, c\}$ with $t = sd$. Let $Z \in \mathbb{P}_E S(f) \mathbb{P}_E$. By b) and Theorem 2.1.9 b) (and Corollary 1.3.7 d)),

\[
Z = \pm (\gamma \tilde{1}_K) Z V'_t = \pm (\gamma \tilde{1}_K) V'_t Z
\]

By Proposition 2.1.2 b),

\[
Z V'_t = \sum_{s \in S} ((Z_{sf}(s, c)) \tilde{1}_K) V'_t \quad + \sum_{s \in S} ((Z_{sf}(sa, c)) \tilde{1}_K) V'_t
\]

\[
+ \sum_{s \in S} ((Z_{sf}(sb, c)) \tilde{1}_K) V'_t \quad + \sum_{s \in S} ((Z_{sf}(sc, c)) \tilde{1}_K) V'_t
\]

\[
V'_t Z = \sum_{s \in S} ((f(c, s)) \tilde{1}_K) V'_t \quad + \sum_{s \in S} ((f(c, sa)) \tilde{1}_K) V'_t
\]

\[
+ \sum_{s \in S} ((f(c, sb)) \tilde{1}_K) V'_t \quad + \sum_{s \in S} ((f(c, sc)) \tilde{1}_K) V'_t
\]

and so by Theorem 2.1.9 a),

\[
Z_e = \pm \gamma f(sc, c) Z_{sc} = \pm \gamma f(sc, c) Z_{sc},
\]

\[
Z_{sc} = \pm \gamma f(s, c) Z_s = \pm \gamma f(s, c) Z_s,
\]

\[
Z_{sa} = \pm \gamma f(sb, c) Z_{sb} = \pm \gamma f(sb, c) Z_{sb},
\]

\[
Z_{sb} = \pm \gamma f(sa, c) Z_{sa} = \pm \gamma f(sa, c) Z_{sa}.
\]
By e), \( Z_{ab} = Z_{ab} = 0 \) for every \( s \in S \). We get (by a), d), and Proposition 2.1.2 b))

\[
\varphi_s \left( 2 \sum_{s} (Z_s \otimes 1_K) V^f_s \right) = \sum_{s} (Z_s \otimes 1_K) V^f_s \pm (\gamma \otimes 1_K) \sum_{s} (Z_s \otimes 1_K) V^f_s = \sum_{s} (Z_s \otimes 1_K) V^f_s + \sum_{s} (Z_s \otimes 1_K) V^f_s = Z.
\]

Thus \( \varphi_s \) is an \( E-\text{C}^* \)-isomorphism with the mentioned inverse.

b) is a long calculation using e).

i) follows from h).

j) follows from Corollary 2.1.17 d).

\[\Box\]

Remark: An example in which the above hypotheses are fulfilled is given in Theorem 4.1.7.

### 2.3 The functor \( \mathcal{S} \)

Throughout this section we assume \( T \) finite.

In this section we present the construction in the frame of category theory. Some of the results still hold for \( T \) locally finite.

**Definition 2.3.1** The above construction of \( S(f) \) can be done for an arbitrary \( E \)-module \( F \), in which case we shall denote the result by \( S(F) \). Moreover we shall write \( V^F_t \) instead of \( V^f_t \) in this case.

If \( F \) is an \( E \)-module then \( S(F) \) is canonically an \( E \)-module. If in addition \( F \) is adapted then \( S(F) \) is adapted and isomorphic to \( S(F, F) \). If \( F \) is an \( E \)-\( \text{C}^* \)-algebra then \( S(F) \) is also an \( E \)-\( \text{C}^* \)-algebra.

**Proposition 2.3.2** If \( F, G \) are \( E \)-modules and \( \varphi : F \to G \) is an \( E \)-linear \( \text{C}^* \)-homomorphism then the map

\[
S(\varphi) : S(F) \to S(G), \quad X \mapsto \sum_{t} ((\varphi X_t) \otimes 1_K) V^G_t
\]

is an \( E \)-linear \( \text{C}^* \)-homomorphism, injective or surjective if \( \varphi \) is so.

The assertion follows from Theorem 2.1.9 a), c), g).

\[\Box\]

**Corollary 2.3.3** Let \( F_1, F_2, F_3 \) be \( E \)-modules and let \( \varphi : F_1 \to F_2, \psi : F_2 \to F_3 \) be \( E \)-linear \( \text{C}^* \)-homomorphisms.

a) \( S(\psi) \circ S(\varphi) = S(\varphi \circ \psi) \).

b) If the sequence

\[
0 \to F_1 \xrightarrow{\varphi} F_2 \xrightarrow{\psi} F_3
\]

is exact then the sequence

\[
0 \to S(F_1) \xrightarrow{S(\varphi)} S(F_2) \xrightarrow{S(\psi)} S(F_3)
\]

is also exact.

c) The covariant functor \( S : \mathcal{M}_E \to \mathcal{M}_E \) is exact.

a) is obvious.

b) Let \( Y \in \text{Ker} S(\psi) \). For every \( t \in T \), \( Y_t \in \text{Ker} \psi = \text{Im} \varphi \). If we identify \( F_1 \) with \( \text{Im} \varphi \) then \( Y_t \in F_1 \). It follows \( Y \in \text{Im} \mathcal{S}(\varphi) \), \( \text{Ker} S(\psi) = \text{Im} S(\varphi) \).

c) follows from b) and Proposition 2.3.2.
COROLLARY 2.3.4 Let $F$ be an adapted $E$-module and put

$$\iota: F \to \tilde{F}, \quad x \mapsto (0, x)$$

$$\pi: \tilde{F} \to E, \quad (\alpha, x) \mapsto \alpha.$$ 

Then the sequence

$$0 \to S(F) \xrightarrow{\delta_i} S(\tilde{F}) \xrightarrow{\delta_i} S(E) \to 0$$

is split exact. \hfill \blacksquare

PROPOSITION 2.3.5 The covariant functor $S: \mathbb{M}_E \to \mathbb{M}_E$ (resp. $S: \mathbb{C}_E \to \mathbb{C}_E^1$) (Proposition 2.3.2, Corollary 2.3.3 a)) is continuous with respect to the inductive limits (Proposition 36 a),b)).

Let $\{(F_i)_{i \in I} \in \mathbb{M}_E \} (\text{resp. } \mathbb{C}_E)$ be an inductive system in the category $\mathbb{M}_E$ (resp. $\mathbb{C}_E$) and let $(F, (\varphi_i)_{i \in I})$ be its limit in the category $\mathbb{M}_E$ (resp. $\mathbb{C}_E$). Then $\{(S(F_i))_{i \in I}, (S(\varphi_i))_{i \in I})\}$ is an inductive system in the category $\mathbb{M}_E$ (resp. $\mathbb{C}_E$). Let $\{G, (\varphi_i)_{i \in I}\}$ be its limit in this category and let $\psi: G \to S(F)$ be the $E$-linear $C^*$-homomorphism such that $\psi(\varphi_i) = S(\varphi_i)$ for every $i \in I$. In the $\mathbb{C}_E$ case, for $\alpha \in E$ and $i \in I$,

$$\psi(\alpha \otimes 1_K) = \psi(\varphi_i)(\alpha \otimes 1_K) = (S(\varphi_i))(\alpha \otimes 1_K) = \alpha \otimes 1_K$$

so that $\psi$ is an $E$-$C^*$-homomorphism.

Let $i \in I$ and let $x \in \text{Ker}(S(\varphi_i))$. Then $\varphi_i x = 0$ for every $t \in T$. Since $T$ is finite, for every $\varepsilon > 0$ there is a $j \in I$, $j \geq i$, with

$$\|\varphi_j x\| < \frac{\varepsilon}{\text{Card} T}$$

for every $t \in T$. Then

$$\|S(\varphi_i)(x)\| = \sum_{t \in T} \|((\varphi_j x) \otimes 1_K) V_t^F\| < \varepsilon.$$ 

It follows

$$\|\psi(x)\| = \inf_{j \in I, j \geq i} \|S(\varphi_j)(x)\| = 0,$$

$$\psi x = 0, \quad x \in \text{Ker}(\psi), \quad \text{Ker}(\psi) \subset \text{Ker}(\psi).$$

By Lemma 1.2.8, $\psi$ is injective. Since

$$\bigcup_{\varphi \in I} \text{Im} S(\varphi) \subset \text{Im} \psi,$$

$\text{Im} \psi$ is dense in $S(F)$. Thus $\psi$ is surjective and so an $E$-$C^*$-isomorphism \hfill \blacksquare

PROPOSITION 2.3.6 Let $\theta: F \to G$ be a surjective morphism in the category $\mathbb{C}_E$. We use the notation of Theorem 2.2.18 and mark with an exponent if this notation is used with respect to $F$ or to $G$. For every $Y \in \text{Im} S(g^F)$, there is a $Z \in S(g^F)$ such that

$$Z^* Z = p_x^F, \quad S(\theta) Z = q_x^F Y.$$ 

By Proposition 2.3.2 c), $S(\theta)$ is surjective and so there is a $Z_0 \in S(g^F)$ with $\|Z_0\| = 1$ and $S(\theta) Z_0 = Y$. Put

$$Z = p_x^F Z_0 + X^F (1 - Z_0^* Z_0)^{1/2}. $$
By Theorem 2.2.18 b),

\[ Z'Z = P_0^\theta Z_0Z_0 + (1-Z_0^\theta Z_0)^\frac{1}{2}(X^\theta)^*X^\theta(1-Z_0^\theta Z_0)^\frac{1}{2} = P_0^\theta Z_0Z_0 + P_0^\theta (1-Z_0^\theta Z_0) = P_0^\theta. \]

Since

\[ S(\theta)(1-Z_0^\theta Z_0) = 1-Y^*Y = 0 \]

We get

\[ S(\theta)(1-Z_0^\theta Z_0)^\frac{1}{2} = 0, \quad S(\theta)Z = P_0^\theta Y = \phi_0^\theta Y. \]

**PROPOSITION 2.3.7** Let F be an adapted E-module and Ω a locally compact space. We define for \( X \in \mathcal{S}(C_0(\Omega, F)) \) (see Corollary 1.2.5 d)) and \( Y \in \mathcal{C}_0(\Omega, \mathcal{S}(F)) \),

\[ \phi X : \Omega \rightarrow \mathcal{S}(F), \quad \omega \mapsto \sum_{t \in T} (X_t(\omega) \otimes 1_K)V_t^F, \]

\[ \psi Y = \sum_{t \in T} (Y(\cdot)_t \otimes 1_K)V_t^{\mathcal{S}(\Omega, F)}. \]

Then

\[ \phi : \mathcal{S}(C_0(\Omega, F)) \rightarrow \mathcal{C}_0(\Omega, \mathcal{S}(F)), \]

\[ \psi : \mathcal{C}_0(\Omega, \mathcal{S}(F)) \rightarrow \mathcal{S}(C_0(\Omega, F)) \]

are \( E \)-linear \( C^* \)-isomorphisms and \( \varphi = \psi^{-1} \).

Let \( \omega_0 \in \Omega \) and assume F is an \( E \)-\( C^* \)-algebra. Then the above maps \( \varphi \) and \( \psi \) induce the following \( E \)-\( C^* \)-isomorphisms

\[ \mathcal{S}(\{ X \in \mathcal{C}_0(\Omega, F) \mid X(\omega_0) \in \mathcal{E} \}) \rightarrow \{ Y \in \mathcal{C}_0(\Omega, \mathcal{S}(F)) \mid Y(\omega_0) \in \mathcal{E}(\mathcal{E}) \}. \]

Let \( X, X' \in \mathcal{S}(C_0(\Omega, F)) \) and \( Y, Y' \in \mathcal{C}_0(\Omega, \mathcal{S}(F)) \). By Proposition 2.1.23 b) and Corollary 2.1.10 a),

\[ \phi X \in \mathcal{C}_0(\Omega, \mathcal{S}(F)), \quad \psi Y \in \mathcal{S}(C_0(\Omega, F)) \]

and it is easy to see that \( \varphi \) and \( \psi \) are \( E \)-linear. By Theorem 2.1.9 c),g), for \( t \in T \) and \( \omega \in \Omega \),

\[ ((\varphi X)'(\omega))_t = \bar{f}(t)((\varphi X)(\omega)_{s,t})^* = \bar{f}(t)X_{s,t}(\omega)^* = (X^*\omega)_t, \]

\[ ((\varphi X)(\varphi X')(\omega))_t = \sum_{s \in T} f(s, s^{-1} t)(\varphi X)(\omega)_s((\varphi X')(\omega))_{s^{-1} t} = \sum_{s \in T} f(s, s^{-1} t)X_sX'_{s^{-1} t}(\omega) = (XX')_t(\omega) = ((\varphi (XX'))(\omega))_t, \]

so

\[ (\varphi X)' = \varphi X^*, \quad (\varphi X)(\varphi X') = \varphi(XX') \]
and \( \varphi \) is a \( C^* \)-homomorphism. Similarly

\[
(\psi Y')_t(\omega) = (Y'(\omega))_t = f(t)(Y(\omega))_t = \bar{f}(t)((\psi Y)_t(\omega))^* = ((\psi Y')^*)_t(\omega),
\]

\[
((\psi Y)(\psi Y'))_t(\omega) = \left( \sum_{s \in T} f(s, s^{-1}t)(\psi Y)_s(\psi Y')_{s^{-1}t} \right)(\omega) = \sum_{s \in T} f(s, s^{-1}t)(\psi Y)_s(\psi Y')_{s^{-1}t}(\omega) = \sum_{s \in T} f(s, s^{-1}t)Y(\omega)_s Y'(\omega)_{s^{-1}t} = (Y(\omega)Y'(\omega))_t = (\psi Y')_t(\omega)
\]

so

\[
\psi Y^* = (\psi Y)^*, \quad (\psi Y)(\psi Y') = \psi(YY')
\]

and \( \varphi \) is a \( C^* \)-homomorphism. Moreover

\[
(\varphi \phi X)_t(\omega) = ((\varphi X)(\omega))_t = X_t(\omega), \quad ((\varphi Y)(\omega))_t = (\psi Y)_t(\omega) = (Y(\omega))_t,
\]

so \( \varphi \psi X = X \) and \( \varphi \psi Y = Y \) which proves the assertion.

The last assertion is easy to see.

**Proposition 2.3.8** Let \( F \) be an adapted \( E \)-module,

\[
0 \rightarrow F \xrightarrow{\iota} F \xrightarrow{\pi} E \rightarrow 0,
\]

\[
0 \rightarrow S(F) \xrightarrow{\alpha} S(F) \xrightarrow{\pi_0} E \rightarrow 0
\]

the associated exact sequences (Proposition 1.2.4 h)), and

\[
\varphi : S(F) \xrightarrow{\alpha} S(E), \quad (\alpha \otimes 1_K) V^E_t,
\]

\[
\psi : S(F) \rightarrow S(F), \quad (\alpha \otimes X) \xrightarrow{\pi_0} S(\iota X + (\alpha \otimes 1_K) V^E_t).
\]

Then \( \varphi \) is an injective \( E \)-\( C^* \)-homomorphism and \( S(\pi) \alpha \varphi = \pi_0 \).

**Proposition 2.3.9** If \( E \) is commutative and \( F \) is an \( E \)-module then the map

\[
\varphi : S(E) \otimes F \rightarrow S(F), \quad X \otimes Y \rightarrow \sum_{t \in T} ((X_t \otimes 1_K) V^E_t)
\]

is a surjective \( C^* \)-homomorphism. If in addition \( E = K \) then \( \varphi \) is a \( C^* \)-isomorphism with inverse

\[
\psi : S(F) \rightarrow S(E) \otimes F, \quad Y \rightarrow \sum_{t \in T} (V^E_t \otimes Y_t).
\]
It is obvious that \( \varphi \) is surjective. For \( X, Y \in S(E) \) and \( x, y \in F \), by Theorem 2.1.9 c),g) and Proposition 2.1.2 b),d),e),

\[
\varphi((X \otimes x)^*) = \varphi(X^* \otimes x^*) = \sum_{t \in T} \left( (X^*)_t x^*_t \otimes 1_K \right) V^f_t = \\
= \sum_{t \in T} \left( (f(t)X_{r^{-1}})^* x^*_t \otimes 1_K \right) V^f_t = \sum_{t \in T} \left( (X_{r^{-1}})^* x^*_t \otimes 1_K \right) (V^f_t)^* = \\
= \sum_{t \in T} \left( (X_t)^* x^*_t \otimes 1_K \right) (V^f_t)^* = (\varphi(X \otimes x))^*,
\]

\[
\varphi(X \otimes x) \varphi(Y \otimes y) = \sum_{s, t \in T} \left( (sX_s X_y y \otimes 1_K) V^f_s V^f_t = \\
= \sum_{s, t \in T} \left( ((XY)^*_t x^*_t \otimes 1_K) V^f_t = \varphi((X \otimes x)(Y \otimes y))
\right.
\]

so \( \varphi \) is a \( C^* \)-homomorphism.

Assume now \( E = K \) and let \( X \in S(E) \) and \( x \in F \). Then

\[
\psi \varphi(X \otimes x) = \psi \sum_{s \in T} \left( (X_s x^*_s \otimes 1_K) V^f_s = \sum_{s \in T} V^f_s \otimes (X_s x) = \\
= \left( \sum_{s \in T} X_s V^f_t \right) \otimes x = X \otimes x
\right.
\]

which proves the last assertion (by using the first assertion).

\[\Box\]

**EXAMPLES**

We draw the reader’s attention to the fact that in additive groups the neutral element is denoted by 0 and not by 1.

### 3.1 \( T \cong \mathbb{Z}_2 \)

**PROPOSITION 3.1.1**

a) The map

\[
\psi : \mathcal{F} (\mathbb{Z}_2, E) \rightarrow Un E^c, \quad f \mapsto f(1, 1)
\]

is a group isomorphism.

b) \( \psi \left( \{ \delta \lambda | \lambda \in \Lambda (\mathbb{Z}_2, E) \} \right) = \{ x^2 | x \in Un E^c \} \).

c) If there is an \( x \in E^c \) with \( x^2 = f(1, 1) \) (in which case \( x \in Un E^c \)) then the map

\[
\varphi : S(f) \rightarrow E \times E, \quad X \mapsto (X_0 + xX_1, X_0 - xX_1)
\]

is an \( E \cdot C^* \)-isomorphism.

d) If \( K = \mathbb{C} \) and if \( A \) is a connected and simply connected compact space or a totally disconnected compact space then for every \( x \in Un C(A) \) there is a \( y \in C(A, \mathbb{C}) \) with \( x = e^y \).

e) Assume \( K = \mathbb{R} \).

e1) There are uniquely \( p, q \in Pr E^c \) with

\[
p + q = 1_E, \quad pf(1, 1) = p, \quad qf(1, 1) = -q.
\]
e2) The map

\[ \varphi : \mathcal{S}(f) \rightarrow (pE) \times (pE) \times \hat{q}E, \quad X \rightarrow \tilde{X}, \]

where \( \hat{q}E \) denotes the complexification of the C*-algebra \( qE \) and

\[ \tilde{X} := (p(X_0 + X_1), p(X_0 - X_1), (qX_0, qX_1)) \]

for every \( X \in \mathcal{S}(f) \), is an \( E-C^* \)-isomorphism. In particular if \( f(1,1) = 1 \) then \( \mathcal{S}(f) \) is isomorphic to the complexification of \( E \).

f) Assume \( \mathbb{K} = \mathbb{C} \), let \( \sigma(E^*) \) be the spectrum of \( E^* \), and let \( f_{11} \) be the function of \( C(\sigma(E^*), \mathbb{C}) \) corresponding to \( f_{11} \) by the Gelfand transform. Then

\[ \{ e^{i\theta} \in \mathbb{R}, e^{2i\theta} \in f_{11}(\sigma(E^*)) \} \]

is the spectrum of \( V_1 \).

a) follows from Proposition 1.1.2 a) (and Proposition 1.1.4 a)).

b) follows from Definition 1.1.3.

c) For \( X, Y \in \mathcal{S}(f) \), by Theorem 2.1.9 c),g) (and Proposition 1.1.2 a)),

\( (X^*)_0 = (X_0)^*, \quad (X^*)_1 = (x^*(X_1))^* \),

\( (XY)_0 = X_0Y_0 + x^2X_1Y_1, \quad (XY)_1 = X_0Y_1 + X_1Y_0, \)

so

\[ \varphi(X^*) = ((X_0)^* + x(x^*)^2(X_1)^* = (X_0)^* + x(x^*)_1(X_1)^* = (\varphi X)^*, \]

\( (\varphi X)(\varphi Y) = ((X_0 + xX_1)(Y_0 + xY_1), (X_0 - xX_1)(Y_0 - xY_1)) = \)
\[ = (X_0Y_0 + xX_0Y_1 + xX_1Y_0 + x^2X_1Y_1, X_0Y_1 - xX_0Y_1 - xX_1Y_0 + x^2X_1Y_1) = \]
\[ = ((XY)_0 + x(XY)_1, (XY)_0 - x(XY)_1) = \varphi(XY) \]

i.e. \( \varphi \) is an \( E-C^* \)-homomorphism. \( \varphi \) is obviously injective.

Let \( (y, z) \in E \times E \). If we take \( X \in \mathcal{S}(f) \) with

\[ X_0 := \frac{1}{2}(y + z), \quad X_1 := \frac{1}{2}x^*(y - z) \]

then \( \varphi X = (y, z) \), i.e. \( \varphi \) is surjective.

d) is known.

e1) follows by using the spectrum of \( E^* \).

e2) Put

\[ \psi : \mathcal{S}(f) \rightarrow \hat{q}E, \quad X \rightarrow (qX_0, qX_1). \]

For \( X, Y \in \mathcal{S}(f) \), by Theorem 2.1.9 c),g),

\[ \psi(X^*) = (q(X^*)_0, q(X^*)_1) = (q(X_0)^*, qf(1,1)^*(X_1)^*) = \]
\[ = ((qX_0)^*, (qX_1)^*) = (\psi X)^*, \]
\[ (\psi X)(\psi Y) = (qX_0, qX_1)(qY_0, qY_1) = \]
\[ = (q(X_0Y_0 - X_1Y_1), (q(X_0Y_1 + X_1Y_0))) = \psi(XY) \]

so \( \psi \) is an \( E-C^* \)-homomorphism. Thus by c), \( \varphi \) is an \( E-C^* \)-homomorphism. The bijectivity of \( \varphi \) is easy to see.
f) By Proposition 2.1.2 e), \( V_1 \) is unitary so its spectrum is contained in \( \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \). For \( \theta \in \mathbb{R} \) and \( X \in S(f) \),

\[
(e^{i\theta} V_0 - V_1) X = X (e^{i\theta} V_1) = (e^{i\theta} X_0 - f_1 X_1) V_0 + ((e^{i\theta} X_1) \otimes 1_k) V_1 - (X_0 \otimes 1_k) V_1 - ((f_1 X_1) \otimes 1_k) V_1 =
\]

\[
= (e^{i\theta} X_0 - f_1 X_1) \otimes 1_k V_0 + ((e^{i\theta} X_1 - X_0) \otimes 1_k) V_1.
\]

Thus \( X \) is the inverse of \( e^{i\theta} V_0 - V_1 \) iff \( X_0 = e^{i\theta} X_1 \) and \( e^{i\theta} X_0 - f_1 X_1 = 1_E \), i.e. \((e^{i\theta} - f_1) \otimes 1_k = 1_E\). Therefore \( e^{i\theta} V_0 - V_1 \) is invertible iff \( e^{i\theta} - f_1 \) does not vanish on \( \sigma(E) \).

**COROLLARY 3.1.2** Assume \( K := \mathbb{R} \) and let \( S \) be a group, \( F \) a unital C*-algebra, \( g \in F(S, F) \), and

\[
h : (S \times \mathbb{Z}_2) \times (S \times \mathbb{Z}_2) \rightarrow Un \ F^c, \quad ((s_1, t_1), (s_2, t_2)) \\
\begin{cases}
g(s_1, s_2) & \text{if } (t_1, t_2) = (1, 1) \\
g(s_1, s_2) & \text{if } (t_1, t_2) \neq (1, 1).
\end{cases}
\]

a) \( h \in F(S \times \mathbb{Z}_2, F) \).

b) \( S(h) = S(g) \), \( \mathsf{S}_{\mathbb{H}}(h) = \mathsf{S}_{\mathbb{H}}(g) \).

Put \( E := \mathbb{R} \) in the above Proposition and define \( f \in F(\mathbb{Z}_2, \mathbb{R}) \) by \( f(1,1) = -1 \) (Proposition 3.1.1 a)). By this Proposition e), \( S(f) \approx \mathbb{C} \).

Thus by Proposition 2.2.11 c), e),

\[
S(h) \approx S(g) \otimes S(f) = \mathsf{S}(g), \quad \mathsf{S}_{\mathbb{H}}(h) \approx \mathsf{S}_{\mathbb{H}}(g) \otimes \mathsf{S}_{\mathbb{H}}(f) = \mathsf{S}_{\mathbb{H}}(g).
\]

**DEFINITION 3.1.3** We put

\[
\mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \}.
\]

**EXAMPLE 3.1.4** Let \( E = \mathbb{C}(\mathbb{T}, \mathbb{C}) \) and \( f \in F(\mathbb{Z}_2, E) \) with

\[
f(1,1) : \mathbb{T} \rightarrow Un \ \mathbb{C}, \quad z \mapsto z.
\]

If we put

\[
\tilde{X} : \mathbb{T} \rightarrow \mathbb{C}, \quad z \mapsto X_0(z^2) + zX_1(z^2)
\]

for every \( X \in S(f) \) then the map

\[
\varphi : S(f) \rightarrow E, \quad X \mapsto \tilde{X}
\]

is an isomorphism of C*-algebras (but not an E-C*-isomorphism).

For \( X, Y \in S(f) \), by Theorem 2.1.9 c), g),

\[
(X^*)_0 = (X_0)^*, \quad (X^*)_1 = f(1,1)(X_1)^*,
\]

\[
(XY)_0 = X_0 Y_0 + f(1,1)X_1 Y_1, \quad (XY)_1 = X_0 Y_1 + X_1 Y_0
\]

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so for $z \in \mathbb{T}$,
\[
\tilde{X}^*(z) = X_0^*(z^2) + zX_1^*(z^2) = X_0(z^2) + zX_1(z^2) = \tilde{X},
\]
\[
(X(z)^*)\tilde{Y}(z) = (X_0(z^2) + zX_1(z^2))(Y_0(z^2) + zY_1(z^2)) =
\]
\[
= X_0(z^2)Y_0(z^2) + zX_0(z^2)Y_1(z^2) + zX_1(z^2)Y_0(z^2) + z^2X_1(z^2)Y_1(z^2) =
\]
\[
(XY)_0(z^2) + z(XY)_1(z^2) = \tilde{XY},
\]
\[
\tilde{X}^* = \tilde{X}, \quad \tilde{XY} = \tilde{XY}.
\]
i.e. $\varphi$ is a $C^*$-homomorphism. If $\varphi X = 0$ then for $z \in \mathbb{T}$,
\[
X_0(z^2) + zX_1(z^2) = 0
\]
so, successively,
\[
X_0(z^2) - zX_1(z^2) = 0, \quad X_0(z^2) = X_1(z^2) = 0, \quad X_0 = X_1 = 0, \quad X = 0
\]
and $\varphi$ is injective.

Put
\[
G := \left\{ \sum_{k \in \mathbb{Z}} c_k z^k \mid (c_k)_{k \in \mathbb{Z}} \in C^*(\mathbb{Z}) \right\} \subset E.
\]

Let
\[
x := \sum_{k \in \mathbb{Z}} c_k z^k \in G
\]
and take $X \in S(f)$ with
\[
X_0 := \sum_{k \in \mathbb{Z}} c_{2k} z^k, \quad X_1 := \sum_{k \in \mathbb{Z}} c_{2k+1} z^k.
\]

Then
\[
\tilde{X} = \sum_{k \in \mathbb{Z}} c_{2k} z^{2k} + z \sum_{k \in \mathbb{Z}} c_{2k+1} z^{2k} = x
\]
so $G \subset \varphi(S(f))$. Since $G$ is dense in $E$, $\varphi(S(f)) = E$ and $\varphi$ is surjective.

**DEFINITION 3.1.5** For every $x \in C(T, \mathbb{C})$ which does not take the value 0 we put
\[
w(x) := \text{winding number of } x := \frac{1}{2\pi i} \int_x \frac{dz}{z} = \frac{1}{2\pi i} \left[ \log x(e^{i\theta}) \right]_{\theta=0}^{\theta=2\pi} \in \mathbb{Z}.
\]

If $A$ is a connected compact space and $\gamma$ is a cycle in $A$ (i.e. a continuous map of $T$ in $A$), which is homologous to 0 (or more generally, if a multiple of $\gamma$ is homologous to 0), then for every $x \in C(A, Un \mathbb{C})$ we have $w(x \circ \gamma) = 0$. If $A$ is a compact space and $x \in C(A, Un \mathbb{C})$ such that $w(x \circ \gamma) = 0$ for every cycle $\gamma$ in $A$ then there is a $y \in C(A, \mathbb{C})$ with $x = e^y$.

**EXAMPLE 3.1.6** Let $E = C(T, \mathbb{C})$, $f \in F(Z_2, E)$, and $n := w(f(1, 1))$.

a) If $n$ is even then there is an $x \in Un E$ with winding number equal to $\frac{n}{2}$ such that the map
\[
S(f) \to E \times E, \quad X \mapsto (X_0 + xX_1, X_0 - xX_1)
\]
is an $E$-$C^*$-isomorphism.

b) If $n$ is odd then $S(f)$ is isomorphic to $E$. 

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c) The group $\mathcal{F}(\mathbb{Z}_2, E)/\mathbb{Z}(\mathbb{Z}_2, E)$ is isomorphic to $\mathbb{Z}_2$ and

$$\text{Card}(\{S(g) \mid g \in \mathcal{F}(\mathbb{Z}_2, E)\}/\cong) = 2.$$ 

d) There is a complex unital $C^*$-algebra $E$ and a family $(f_\beta)_{\beta \in \Psi(\mathbb{N})}$ in $\mathcal{F}(\mathbb{Z}_2, E)$ such that for distinct $\beta, \gamma \in \Psi(\mathbb{N})$, $S(f_\beta) \neq S(f_\gamma)$.

Put

$$\alpha : \mathbb{T} \to \text{Un } \mathbb{C}, \quad z \mapsto z.$$ 

Since $w(f(1,1)x^{-n}) = 0$, there is a $y \in \text{Un } E$ with $w(y) = 0$ and $f(1,1)x^{-n} = y^x$.

a) If we put $x=y\alpha^2$ then $w(x) = \frac{2}{\pi}$ and $f(1,1) = x^2$ and the assertion follows from Proposition 3.1.1 c).

b) We put $x=ya^{-2}$. Then $f(1,1) = ax^2$. Take $g \in \mathcal{F}(\mathbb{Z}_2, E)$ with $g(1,1) = a$ and $\lambda \in \Lambda(\mathbb{Z}_2, E)$ with $(\delta \lambda)(1,1) = x^2$ (Proposition 3.1.1 a),b)). Then $f = g\delta \lambda$. By Example 3.1.4, $S(g)$ is isomorphic to $E$ and by Proposition 2.2.2 a) $\Rightarrow a_2$, $S(f)$ is also isomorphic to $E$.

c) follows from Proposition 3.1.1 b) and Proposition 2.2.2 a,c).

d) Denote by $E$ the $C^*$-direct product of the sequence $(\mathcal{C}(\mathbb{T}, \mathbb{C}_{n,n}))_{\mathbb{N} \in \mathbb{N}}$ and for every $\beta \in \{0,1\}^\mathbb{N}$ define $f_\beta \in \mathcal{F}(\mathbb{Z}_2, E)$ by

$$f_\beta(1,1) : \mathbb{N} \to \text{Un } \mathbb{E}^c, \quad m \mapsto \mathbb{e}_m \mathbb{1}_{c_{n,n}}.$$ 

By a) and b), for distinct $\beta, \gamma \in \{0,1\}^\mathbb{N}$, $S(f_\beta) \neq S(f_\gamma)$ (Proposition 2.1.26 a)).

**EXAMPLE 3.1.7** Let $I, J$ be finite disjoint sets and for all $i \in I \cup J$ and $j \in J$ put $A_i = B_j = \mathbb{T}$. We define the compact spaces $A$ and $B$ in the following way. For $\alpha$ we take first the disjoint union of the spaces $A_i$ for all $i \in I \cup J$ and identify then the points $1 \in A_i$ for all $i \in I \cup J$. For $B$ we take first the disjoint union of all the spaces $B_j$ for all $j \in J$ and identify first the points $1 \in A_i$ for all $i \in I \cup J$ and then also the points $-1 \in A_i$ for all $i \in I$ and $1 \in B_j$ for all $j \in J$.

Let $E := \mathcal{C}(A, \mathbb{C})$ and $f \in \mathcal{F}(\mathbb{Z}_2, E)$ with

$$f(1,1) : A \to \text{Un } \mathbb{C}, \quad z \mapsto \begin{cases} z & \text{if } z \in A_i \text{ with } i \in I \\ 1 & \text{if } z \in A_i \text{ with } i \in J \end{cases}.$$ 

For every $X \in S(f)$ define $\tilde{X} \in \mathcal{C}(B, \mathbb{C})$ by

$$\tilde{X} : B \to \mathbb{C}, \quad z \mapsto \begin{cases} X_0(z^2) + 2X_1(z^2) & \text{if } z \in A_i \text{ with } i \in I \\ X_0(z) + X_1(z) & \text{if } z \in A_i \text{ with } i \in J \\ X_0(z) - X_1(z) & \text{if } z \in B_j \text{ with } j \in J \end{cases}.$$ 

Then the map

$$\varphi : S(f) \to \mathcal{C}(B, \mathbb{C}), \quad X \mapsto \tilde{X}$$ 

is an isomorphism of $C^*$-algebras.

Let $X, Y \in S(f)$. By Theorem 2.1.9 c), g),

$$(X^*,)_0 = (X_0)^*, \quad (X^*)_1 = f(1,1)(X_1)^*.$$ 

$$(XY)_0 = X_0 Y_0 + f(1,1)X_1 Y_1, \quad (XY)_1 = X_0 Y_1 + X_1 Y_0.$$ 

For $z \in A_i$ with $i \in I$,

$$\tilde{X}^*(z) = (X_0)_0(z^2) + z(X^*)_1(z^2) = \overline{X_0(z^2)} + zX_1(z^2) = X_0(z^2)^* + zX_1(z^2) = (\tilde{X})^*(z),$$

$$\tilde{X}(z) Y(z) = (X_0(z)) \overline{Y_0(z^2)} + zX_1(z^2)Y_1(z^2) + \overline{X_0(z^2)} Y_0(z^2) + zX_1(z^2) =$$

$$= (XY)_0 z + zX_0(z^2) Y_1(z^2) + zX_1(z^2) Y_0(z^2) + zX_1(z^2) Y_1(z^2) = (XY)^*(z^2) + z(XY^*)_1(z^2) = \overline{\tilde{Y}(z)}.$$
For \( z \in A_j \) or \( z \in B_j \) with \( j \in J \),

\[
\widetilde{X}(z) = (X^*)_0(z) + (X^*)_1(z) = X_0(z) + X_1(z) = \widetilde{X}(z),
\]

\[
\widetilde{X}(z) \widetilde{Y}(z) = (X_0(z) + X_1(z))(Y_0(z) + Y_1(z)) =
\]

\[
= X_0(z)Y_0(z) + X_0(z)Y_1(z) + X_1(z)Y_0(z) + X_1(z)Y_1(z) =
\]

\[
= (XY)_0(z) + (XY)_1(z) = \widetilde{XY}(z).
\]

Thus \( \varphi \) is a \( \mathbb{C}^* \)-homomorphism. Assume \( \widetilde{X} = 0 \). For \( z \in A_i \) with \( i \in I \),

\[
X_0(z^2) + zX_1(z^2) = 0
\]

so, successively,

\[
X_0(z^2) - zX_1(z^2) = 0, \quad X_0(z^2) = X_1(z^2) = 0, \quad X(z) = 0.
\]

For \( z \in A_j \) with \( j \in J \),

\[
\begin{cases}
X_0(z) + X_1(z) = 0 \\
X_0(z) - X_1(z) = 0,
\end{cases}
\]

so

\[
X_0(z) = X_1(z) = 0, \quad X(z) = 0.
\]

Thus \( \varphi \) is injective.

Let \( x \in \mathbb{C}(B, \mathbb{C}) \) such that for every \( i \in I \) there is a family \( (c_{i,k})_{k \in \mathbb{Z}} \in \mathbb{C}^{(\mathbb{Z})} \) with

\[
x(z) = \sum_{k \in \mathbb{Z}} c_{i,k} z^k
\]

for all \( z \in A_i \). Define \( X_0, X_1 \in E \) in the following way. If \( z \in A_i \) with \( i \in I \) we put

\[
X_0(z) := \sum_{k \in \mathbb{Z}} c_{i,2k} z^k, \quad X_1(z) := \sum_{k \in \mathbb{Z}} c_{i,2k+1} z^k.
\]

If \( z \in A_j \) with \( j \in J \) then we put \( z' := z \in B_j \).

\[
X_0(z) := \frac{1}{2} (x(z) + x(z')), \quad X_1(z) := \frac{1}{2} (x(z) - x(z')).
\]

It is easy to see that \( X_0 \) and \( X_1 \) are well defined. Then

\[
\widetilde{X}(z) = \sum_{k \in \mathbb{Z}} c_{i,2k} z^{2k} + z \sum_{k \in \mathbb{Z}} c_{i,2k+1} z^{2k} = x(z)
\]

for all \( z \in A_i \) with \( i \in I \) and \( \widetilde{X}(z) = x(z) \) for all \( z \cup A_j \in B_j \) with \( j \in J \). Since the elements \( x \) of the above form are dense in \( \mathbb{C}(B, \mathbb{C}) \), \( \varphi \) is surjective.

**EXAMPLE 3.18** Let \( E := \mathbb{C}(\mathbb{T}^2, \mathbb{C}) \) and \( f, g \in \mathcal{F}(\mathbb{Z}_2, E) \) with

\[
\begin{cases}
f(1, 1) : \; \mathbb{T}^2 \to \mathbb{Un} \; \mathbb{C}, \; (z_1, z_2) \mapsto z_1 \\
g(1, 1) : \; \mathbb{T}^2 \to \mathbb{Un} \; \mathbb{C}, \; (z_1, z_2) \mapsto z_2.
\end{cases}
\]

Then the maps

\[
\begin{cases}
S(f) \to E, \quad X \mapsto X_0(z_1^2, z_2) + z_1X_1(z_1^2, z_2) \\
S(g) \to E, \quad X \mapsto X_0(z_1, z_2^2) + z_2X_1(z_1, z_2^2)
\end{cases}
\]

are isomorphisms of \( \mathbb{C}^* \)-algebras.

**Remark.** \( S(f) \) and \( S(g) \) are isomorphic but not \( E-\mathbb{C}^* \)-isomorphic.
**Example 3.19** Let \( E = C(\mathbb{T}^2, \mathbb{C}) \) and \( f \in \mathcal{F}(\mathbb{Z}_2, E) \) with
\[
 f(1, 1) : \mathbb{T}^2 \to \text{Un C}, \quad (z_1, z_2) \mapsto z_1 z_2.
\]
If we put
\[
 \tilde{X} : \mathbb{T}^2 \to \mathbb{C}, \quad (z_1, z_2) \mapsto X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2)
\]
for every \( X \in S(f) \) then the map
\[
 \varphi : S(f) \to E, \quad X \mapsto \tilde{X}
\]
is an injective unital \( C^* \)-homomorphism with
\[
 \varphi(S(f)) = \mathcal{G} = \{ x \in E \mid (z_1, z_2) \in \mathbb{T}^2 \implies x(z_1, z_2) = x(-z_1, -z_2) \}.
\]

In particular \( S(f) \) is isomorphic to \( E \).

Let \( X, Y \in S(f) \). By Theorem 2.1.9 c), g),
\[
 (X^*)_0 = (X_0)^*, \quad (X^*)_1 = f(1, 1)(X_1)^*,
\]
\[
 (XY)_0 = X_0 Y_0 + f(1, 1)X_1 Y_1, \quad (XY)_1 = X_0 Y_1 + X_1 Y_0
\]
so for \( (z_1, z_2) \in \mathbb{T}^2 \),
\[
 \tilde{X}^*(z_1, z_2) = X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2) =
\]
\[
 = X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2) = \tilde{X}(z_1, z_2),
\]
\[
 (X(z_1, z_2))(Y(z_1, z_2)) =
\]
\[
 = (X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2))(Y_0(z_1^2, z_2^2) + z_1 z_2 Y_1(z_1^2, z_2^2)) =
\]
\[
 = X_0(z_1^2, z_2^2)Y_0(z_1^2, z_2^2) + z_1 z_2 X_0(z_1^2, z_2^2)Y_1(z_1^2, z_2^2) +
\]
\[
 + z_1 z_2 X_1(z_1^2, z_2^2)Y_0(z_1^2, z_2^2) + z_1 z_2 X_0(z_1^2, z_2^2)Y_1(z_1^2, z_2^2) =
\]
\[
 = (X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2))(Y_0(z_1^2, z_2^2) + z_1 z_2 Y_1(z_1^2, z_2^2)) = \tilde{X} \tilde{Y} (z_1, z_2),
\]
i.e. \( \varphi \) is a unital \( C^* \)-homomorphism. If \( \tilde{X} = 0 \) then for \( (z_1, z_2) \in \mathbb{T}^2 \),
\[
 X_0(z_1^2, z_2^2) + z_1 z_2 X_1(z_1^2, z_2^2) = 0
\]
so, successively,
\[
 X_0(z_1^2, z_2^2) = 0, \quad X_0(z_1^2, z_2^2) = 0, \quad X_0 = 0, \quad X = 0
\]
and \( \varphi \) is injective.

The inclusion \( S(f) \subseteq \mathcal{G} \) is obvious. Let \((a_{j, k})_{j, k \in \mathbb{Z}}, (b_{j, k})_{j, k \in \mathbb{Z}} \in \mathbb{C}^{(2 \times 2)}\) and
\[
 x = \sum_{j, k \in \mathbb{Z}} a_{j, k} z_1^{2j} z_2^{2k} + \sum_{j, k \in \mathbb{Z}} b_{j, k} z_1^{2j+1} z_2^{2k+1} \in \mathcal{G}.
\]
Define
\[
 X_0 := \sum_{j \in \mathbb{Z}} a_{j, k} z_1^{2j} z_2^{2k}, \quad X_1 := \sum_{j \in \mathbb{Z}} b_{j, k} z_1^{2j} z_2^{2k}.
\]
Then \( \tilde{X} = x \). Since the elements of the above form are dense in \( \mathcal{G} \), \( \varphi(S(f)) = \mathcal{G} \).
If we consider the equivalence relation ~ on \( T^2 \) defined by
\[
(z_1, z_2) \sim (w_1, w_2) : \iff z_1 = -w_1, \quad z_2 = -w_2
\]
then the quotient space \( T^2/\sim \) is homeomorphic to \( T^2 \). Thus \( S(f) \) is isomorphic to \( E \).

**EXAMPLE 3.1.10** Let \( E = C(T^2, \mathbb{C}) \).

a) For \( x \in \text{Un } E \) and \( z \in T \), \( w(x(z)) \) and \( w(x(z, .)) \) do not depend on \( z \), where \( w \) denotes the winding number (Definition 3.1.5).

b) If \( x \in \text{Un } E \) and if
\[
w(x(z, 1)) = w(x(1, z)) = 0
\]
then there is a \( y \in \text{Un } E \) with \( x = y^2 \).

c) Let \( f \in F(Z_2, E) \) and put
\[
\alpha : T \to T^2, \quad z \mapsto (z, 1), \quad \beta : T \to T^2, \quad z \mapsto (1, z),
\]
\[
m = w(f(1) \circ \alpha) \quad n = w(f(1) \circ \beta).
\]

\[c_1\] If \( m + n \) is odd then \( S(f) \) is isomorphic to \( E \).

\[c_2\] If \( m \) and \( n \) are even then \( S(f) \) is isomorphic to \( E \times E \).

\[c_3\] If \( m \) and \( n \) are odd then \( S(f) \) is isomorphic to \( E \).

d) The group \( F(Z_2, E) \) is isomorphic to \( Z_2 \times Z_2 \) and
\[
\text{Card}(\{ S(f) \mid f \in F(Z_2, E) \}/z_5) = 4.
\]

a) follows by continuity.

b) follows from a).

c) Let \( g \in F(Z_2, E) \) with
\[
g(1, 1) : T^2 \to \text{Un } C, \quad (z_1, z_2) \mapsto z_1^n z_2^m.
\]

Then
\[
w(g(1) \circ \alpha) = m, \quad w(g(1) \circ \beta) = n.
\]

By b), there is an \( x \in \text{Un } E \) with \( f(1, 1) = x^2 g(1, 1) \). By Proposition 3.1.1 b) and Proposition 2.2.2 \( a_1 \Rightarrow a_2 \), \( S(f) \approx S(g) \).

\[c_1\] Assume \( m \) even and put
\[
y : T^2 \to \text{Un } C, \quad (z_1, z_2) \mapsto z_1^{m/2} z_2^{n/2}.
\]

If \( h \in F(Z_2, E) \) with
\[
h(1, 1) : T^2 \to \text{Un } C, \quad (z_1, z_2) \mapsto z_2
\]
then \( g(1, 1) = y^2 h(1, 1) \). By Proposition 3.1.1 b) and Proposition 2.2.2 \( a_1 \Rightarrow a_2 \), \( S(g) \approx S(h) \) and by Example 3.1.8 \( a_1 \Rightarrow a_2 \), \( S(h) \approx E \).

Thus \( S(f) \approx E \).

\[c_2\] If we put
\[
y : T^2 \to \text{Un } C, \quad (z_1, z_2) \mapsto z_1^{m/2} z_2^{n/2},
\]
then \( g(1, 1) = y^2 \) and the assertion follows from Proposition 3.1.1 c).

\[c_3\] We put
\[
y : T^2 \to \text{Un } C, \quad (z_1, z_2) \mapsto z_1^{m/2} z_2^{n/2}.
\]
and take \( h \in \mathcal{F}(\mathbb{Z}_2, E) \) with

\[
    h(1, 1) : \mathbb{T}^2 \rightarrow \text{Un } C, \quad (z_1, z_2) \mapsto z_1 z_2
\]

then \( g(1,1) = y^2 h(1,1) \) so by Proposition 3.1.1 b) and Proposition 2.2.2 \( a_1 \Rightarrow a_2, S(g) \approx S(h) \). By Example 3.1.9 \( S(h) \approx E \), so \( S(f) \approx E \).

d) follows from b), Proposition 3.1.1 b), and Proposition 2.2.2 a), c).

**Remark.** In a similar way it is possible to show that for every \( n \in \mathbb{N} \), \( \mathcal{F}(\mathbb{Z}_2, \mathbb{T}^n) / \Lambda(\mathbb{Z}_2, \mathbb{T}^n) \) is isomorphic to \( (\mathbb{Z}_2)^n \) and

\[
    \text{Card}(\{ S(f) \mid f \in \mathcal{F}(\mathbb{Z}_2, \mathbb{T}^n) \}) = 2^n.
\]

**Example 3.1.11** Let \( I, J, K \) be finite pairwise disjoint sets and for every \( i \in I \cup J \cup K \) put \( A_i := B_k := \mathbb{T}^2 \). We define the compact spaces \( A \) and \( B \) in the following way. For \( A \) we take first the disjoint union of the spaces \( A_i \) with \( i \) and is an isomorphism of \( C^* \)-algebras.

**Example 3.1.12** If \( n \in \mathbb{N} \), \( E := \mathcal{C}(\mathbb{T}^n, \mathbb{C}) \), and \( f \in \mathcal{F}(\mathbb{Z}_2, \mathcal{C}(\mathbb{T}^n, \mathbb{C})) \) then \( S(f) \) is isomorphic either to \( \mathcal{C}(\mathbb{T}^n, \mathbb{C}) \) or to \( \mathcal{C}(\mathbb{T}^n, \mathbb{C}) \times \mathcal{C}(\mathbb{T}^n, \mathbb{C}) \).

**Example 3.1.13** Assume \( E := \mathcal{C}(A, C) \), where \( A \) denotes Möbius’s band (resp. Klein’s bottle), i.e. the topological space obtained from \([0,2\pi] \times [-\pi, \pi] \) by identifying the points \((0, \alpha) \) and \((2\pi, -\alpha) \) for all \( \alpha \in [-\pi, \pi] \) (resp. and the points \((\theta, -\pi) \) and \((\theta, \pi) \) for all \( \theta \in [0, 2\pi] \)). We put \( B := \{ -\pi, \pi \} \) (resp. \( B := \mathbb{T}^2 \)) and

\[
    \bar{x} : [0, 2\pi] \times [-\pi, \pi] \rightarrow \mathbb{C}, \quad (\theta, \alpha) \rightarrow \left\{ \begin{array}{ll}
    x(2\theta, \alpha) & \text{if } \theta \in [0, \pi] \\
    x(2(\theta-\pi), -\alpha) & \text{if } \theta \in [\pi, 2\pi]
    \end{array} \right.
\]

for every \( x \in E \).

a) \( \bar{x} \) is well-defined and belongs to \( \mathcal{C}(B, C) \) for every \( x \in E \).

b) If \( f_{\alpha \theta}(0, \alpha) = e^{i\theta} \) for all \( (0, \alpha) \in [0, 2\pi] \times [-\pi, \pi] \) then the map

\[
    \psi : S(f) \rightarrow C(B, \mathbb{C}), \quad X \mapsto \bar{x}_0 + e^{i\theta} \bar{x}_1
\]

is a \( C^* \)-isomorphism.

c) Let \( x \in \text{Un } E \). If \( w(x, 0) = 0 \) (where \( w \) denotes the winding number) then there is a \( y \in E \) with \( e^{i\theta} = x \).
d) Let \( x \in \text{Un} E \) and put \( n := w(x, 0) \). Then there is a \( y \in E \) with \( e^y = e^{-in} x \).

e) The group \( \mathcal{F}(\mathbb{Z}_2, A)/\mathcal{H}(\mathbb{Z}_2, A) \) is isomorphic to \( \mathbb{Z}_2 \).

f) If \( w(f_1, 0) \) is even (resp. odd) then \( S(f) \) is isomorphic to \( E \times E \) (resp. to \( C(B, \mathbb{C}) \)).

a) For \( \alpha \in [-\pi, \pi] \),

\[
\bar{x}(\pi, \alpha) = x(2\pi, \alpha) = x(0, -\alpha) = \bar{x}(\pi, \alpha)
\]

so \( \bar{x} \) is well-defined. Moreover

\[
\bar{x}(0, \alpha) = x(0, \alpha) = x(2\pi, -\alpha) = \bar{x}(2\pi, \alpha)
\]

and in the case of Klein’s bottle

\[
\begin{align*}
\bar{x}(\theta, -\pi) &= x(2\theta, -\pi) = x(2\theta, \pi) = \bar{x}(\theta, \pi) & \text{if } & \theta \in [0, \pi] \\
\bar{x}(\theta, -\pi) &= x(2(\theta-\pi), \pi) = x(2(\theta-\pi), -\pi) = \bar{x}(\theta, \pi) & \text{if } & \theta \in [\pi, 2\pi]
\end{align*}
\]

i.e. \( \bar{x} \in C(B, \mathbb{C}) \).

b) For \( X, Y \in S(f) \) and \( (\theta, \alpha) \in [0,2\pi] \times [-\pi, \pi] \), by Theorem 2.19 c), g),

\[
(q\phi X^*)(\theta, \alpha) = (\bar{X}^*)_1 (\theta, \alpha) + e^{i\theta} (\bar{X}^*)_1 (\theta, \alpha) =
\]

\[
= (\bar{X}_0)^*(\theta, \alpha) + e^{i\theta} (\bar{X}_0^2 (\theta, \alpha))
\]

\[
= \begin{cases} 
X_0(2\theta, \alpha) + e^{i\theta} (X_0^2 (2\theta, \alpha)) & \text{if } \theta \in [0, \pi] \\
X_0(2(\theta-\pi), -\alpha) + e^{i\theta} (X_1^2 (2(\theta-\pi), -\alpha)) & \text{if } \theta \in [\pi, 2\pi]
\end{cases}
\]

\[
(q\phi X)(q\phi Y) = (\bar{X}_0 + e^{i\theta} \bar{X}_1) (\bar{Y}_0 + e^{i\theta} \bar{Y}_1) = \bar{X}_0 \bar{Y}_0 + e^{i\theta} \bar{X}_0 \bar{Y}_1 + e^{i\theta} \bar{X}_1 \bar{Y}_0 + e^{2i\theta} \bar{X}_1 \bar{Y}_1,
\]

\[
= \bar{X}_0 \bar{Y}_0 + e^{2i\theta} \bar{X}_1 \bar{Y}_1 + e^{i\theta} (\bar{X}_0 \bar{Y}_1 + \bar{X}_1 \bar{Y}_0) = (q\phi X)(q\phi Y),
\]

i.e. \( \phi \) is a \( C^* \)-homomorphism. If \( \phi X = 0 \) then for \( \alpha \in [-\pi, \pi] \),

\[
\begin{cases} 
X_0(2\theta, \alpha) + e^{i\theta} X_1(2\theta, \alpha) = 0 & \text{if } \theta \in [0, \pi] \\
X_0(2(\theta-\pi), -\alpha) + e^{i\theta} X_1(2(\theta-\pi), -\alpha) = 0 & \text{if } \theta \in [\pi, 2\pi]
\end{cases}
\]

so for \( \theta \in [0, \pi] \), replacing \( \theta \) by \( \theta + \pi \) and \( \alpha \) by \( -\alpha \) in the second relation,

\[
X_0(2\theta, \alpha) - e^{i\theta} X_1(2\theta, \alpha) = 0.
\]

It follows successively

\[
X_0(2\theta, \alpha) = X_1(2\theta, \alpha) = 0,
\]

Thus \( \phi \) is injective.
Let $y \in C(B, \mathbb{C})$. Put

$$
\begin{cases}
X_0 : [0, 2\pi] \times [-\pi, \pi] \to \mathbb{C}, & (\theta, \alpha) \mapsto \frac{1}{2} \left( y \left( \frac{\theta}{2}, \alpha \right) + y \left( \frac{\theta}{2} + \pi, -\alpha \right) \right) \\
X_1 : [0, 2\pi] \times [-\pi, \pi] \to \mathbb{C}, & (\theta, \alpha) \mapsto \frac{1}{2} e^{-i\theta} \left( y \left( \frac{\theta}{2}, \alpha \right) - y \left( \frac{\theta}{2} + \pi, -\alpha \right) \right)
\end{cases}
$$

For $\alpha \in [-\pi, \pi]$,

$$
\begin{align*}
X_0(0, \alpha) &= \frac{1}{2} (y(0, \alpha) + y(\pi, -\alpha)) \\
X_0(2\pi, -\alpha) &= \frac{1}{2} (y(\pi, -\alpha) + y(2\pi, \alpha)) \\
X_1(0, \alpha) &= \frac{1}{2} (y(0, \alpha) - y(\pi, -\alpha)) \\
X_1(2\pi, -\alpha) &= -\frac{1}{2} (y(\pi, -\alpha) - y(2\pi, \alpha))
\end{align*}
$$

so $X_0, X_1 \in E$. Moreover for $(\theta, \alpha) \in [0, 2\pi] \times [-\pi, \pi]$, \[
\tilde{X}_0(\theta, \alpha) + e^{i\theta} \tilde{X}_1(\theta, \alpha) = \\
= \begin{cases}
X_0(2\theta, \alpha) + e^{i\theta} X_1(2\theta, \alpha) & \text{if } \theta \in [0, \pi] \\
X_0(2(\theta-\pi), -\alpha) + e^{i\theta} X_1(2(\theta-\pi), -\alpha) & \text{if } \theta \in [\pi, 2\pi]
\end{cases} = \\
= \begin{cases}
\frac{1}{2} (y(\theta, \alpha) + y(\theta + \pi, -\alpha) + y(\theta, -\alpha) - y(\theta + \pi, -\alpha)) & \text{if } \theta \in [0, \pi] \\
\frac{1}{2} (y(\theta-\pi, -\alpha) + y(\theta, \alpha) - y(\theta-\pi, -\alpha) + y(\theta, \alpha)) & \text{if } \theta \in [\pi, 2\pi]
\end{cases} = y(\theta, \alpha)
\]

t.e. \(\varphi\) is surjective.

c) If \(A\) is M"{o}bius’s band then the assertion is obvious so assume \(A\) is Klein’s bottle. The winding numbers of

$$
\begin{cases}
[0, 2\pi] \to \mathbb{C}, & \alpha \mapsto x(0, \alpha) \\
[0, 2\pi] \to \mathbb{C}, & \alpha \mapsto x(2\pi, \alpha)
\end{cases}
$$

are equal by homotopy, but their sum is equal to 0. Thus these winding numbers are equal to 0. The paths $\theta$ and $\alpha$ on $A$ generate the homotopy group of $A$. Thus the winding number of $x$ on any path of $A$ is 0 and the assertion follows.

d) The winding number of

$$
[0, 2\pi] \to \mathbb{C}, \quad \theta \mapsto e^{-i\varphi} x(\theta, 0)
$$

is 0 and the assertion follows from c).

e) The assertion follows from d) and Proposition 3.1.1 b).

f) The assertion follows from b), d), Proposition 2.2.2 \(a_1 \Rightarrow a_2\), and Proposition 3.1.1 c).

3.2 \(T := \mathbb{Z}_2 \times \mathbb{Z}_2\)

PROPOSITION 3.2.1 Let $E$ be a unital $C^*$-algebra and let $a, b, c$ be the three elements of $(\mathbb{Z}_2 \times \mathbb{Z}_2) \setminus \{(0, 0)\}$. Put

$$
A = \{(\alpha, \beta, y, z) \in (Un \ E)^4 \mid z^2 = 1_e\}
$$

\[\text{a}\]

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and for every \( q \in A \) and \( \sigma \in (Un E^r)^3 \) denote by \( f_q \) and \( g_\sigma \) the functions defined by the following tables:

<table>
<thead>
<tr>
<th>( f_q )</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \beta \gamma )</td>
<td>( \gamma )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>b</td>
<td>( \epsilon \gamma )</td>
<td>( \epsilon \gamma )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>c</td>
<td>( \epsilon \beta )</td>
<td>( \epsilon \alpha )</td>
<td>( \alpha \beta )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( g_\sigma )</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \sigma \alpha^2 )</td>
<td>( \alpha \beta \gamma )</td>
<td>( \alpha \beta \gamma )</td>
</tr>
<tr>
<td>b</td>
<td>( \alpha \beta \gamma )</td>
<td>( \beta ^2 )</td>
<td>( \beta \gamma )</td>
</tr>
<tr>
<td>c</td>
<td>( \alpha \beta \gamma )</td>
<td>( \beta \gamma )</td>
<td>( \gamma ^2 )</td>
</tr>
</tbody>
</table>

a) \( f_q \in \mathcal{F}(\mathbb{Z}_2 \times \mathbb{Z}_2, E) \) for every \( q \in A \) and the map

\[
A \longrightarrow \mathcal{F}(\mathbb{Z}_2 \times \mathbb{Z}_2, E), \quad \phi \mapsto f_\phi
\]

is bijective.

b) \( g_\sigma \in \{ \delta \lambda | \lambda \in \Lambda(\mathbb{Z}_2 \times \mathbb{Z}_2, E) \} \) for every \( \sigma \in (Un E^r)^3 \) and the map

\[
(\text{Un } E^r)^3 \longrightarrow \{ \delta \lambda | \lambda \in \Lambda(\mathbb{Z}_2 \times \mathbb{Z}_2, E) \}, \quad \sigma \mapsto g_\sigma
\]

is bijective.

c) The following are equivalent for all \( q := (\alpha, \beta, \gamma, \epsilon) \in A \) and \( q' := (\alpha', \beta', \gamma', \epsilon') \in A:\n
c_1) \( S_q(f) \equiv S_{q'}(f) \).

c_2) \( \epsilon = \epsilon' \) and there are \( x, y, z \in \text{Un } E^r \) with

\[
x^2 = \beta \beta'^* \gamma \gamma'^*, \quad y^2 = \alpha \alpha'^* \gamma \gamma'^*, \quad z^2 = \alpha \alpha'^* \beta \beta'^*.
\]

c_3) \( \epsilon = \epsilon' \) and there are \( x, y \in \text{Un } E^r \) with

\[
x^2 = \beta \beta'^* \gamma \gamma'^*, \quad y^2 = \alpha \alpha'^* \gamma \gamma'^*.
\]

d) The following are equivalent for all \( q := (\alpha, \beta, \gamma, \epsilon) \in A \) and \( X \in \mathcal{S}(f_q) \):

d_1) \( \lambda \epsilon \in \{ \phi \mid \phi \in \mathbb{Z}_2 \times \mathbb{Z}_2 \} \).

d_2) \( t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( X_t = X_c \).

e) The following are equivalent for all \( q := (\alpha, \beta, \gamma, \epsilon) \in A \) and \( X \in \mathcal{S}(f_q) \):

e_1) \( X \in \mathcal{S}(f_q) \).

e_2) \( t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( X_t = X_c \in \text{Un } E^r \).

f) \( \sigma := (\alpha, \beta, \gamma, \epsilon) \in A \) and \( X, Y \in \mathcal{S}(f_q) \).

\[
(X^r)^0 = X_0^r, \quad (X^r)^a = \beta \gamma X_0^r, \quad (X^r)^b = \epsilon \alpha X_0^r, \quad (X^r)^c = \alpha \beta X_0^r,
\]
\[
(XX)^a = X_0 Y_0 + \beta \gamma X_0 Y_0 + \epsilon \alpha X_0 Y_0 + \alpha \beta X_0 Y_0,
\]
\[
(XX)^b = X_0 Y_0 + \beta \gamma X_0 Y_0 + \epsilon \alpha X_0 Y_0 + \alpha \beta X_0 Y_0,
\]
\[
(XX)^c = X_0 Y_0 + \gamma X_0 Y_0 + \epsilon \alpha X_0 Y_0 + \alpha \beta X_0 Y_0.
\]

g) Assume \( \mathbb{K} = \mathbb{C} \), let \( \sigma(E^r) \) be the spectrum of \( E^r \), and for every \( \delta \in E^r \) let \( \tilde{\delta} \) be its Gelfand transform. Then

\[
\sigma(V_0) = \{ \epsilon^{\phi \delta} | \phi, \delta \in \mathbb{R}, \phi \in \mathbb{R} \},
\]
\[
\sigma(V_b) = \{ \epsilon^{2\phi \delta \gamma} | \phi, \delta \in \mathbb{R}, \phi \in \mathbb{R} \},
\]
\[
\sigma(V_c) = \{ \epsilon^{\phi \delta \epsilon} | \phi, \delta \in \mathbb{R}, \phi \in \mathbb{R} \}.
\]

a) is a long calculation.

b) is easy to verify.

c_1 \Rightarrow c_2 \text{ By Proposition 2.2.2 } a_2 \Rightarrow a_1 \text{ there is a } \lambda \in \Lambda(\mathbb{Z}_2 \times \mathbb{Z}_2, E) \text{ with } f_\lambda = f_\phi \delta \lambda. \text{ By b), there is a } \sigma := (x, y, z) \in (\text{Un } E^r)^3 \text{ with } f_\sigma = f_\phi g_\sigma. \text{ We get } \epsilon = \epsilon' \text{ and }

\[
\alpha \alpha'^* = x \ast yz, \quad \beta \beta'^* = xy'z, \quad \gamma \gamma'^* = xyz + .
\]
It follows $xyz = αα^* ββ^* γγ^*$ so

$$x^2 = ββ^* γγ^*, \quad y^2 = αα^* γγ^*, \quad z^2 = αα^* ββ^*.$$  

$c_2 \Rightarrow c_4$ is trivial.

$c_3 \Rightarrow c_2$ If we put $z := xyy^*$ then

$$z^2 = ββ^* γγ^* αα^* γγ^* γγ^* = αα^* ββ^*.$$  

$c_2 \Rightarrow c_1$ follows from b) and Proposition 2.2.2 $a_1 \Rightarrow a_2$.

d) follows from Corollary 2.1.24 b).

e) follows from Corollary 2.1.24 c).

f) follows from Theorem 2.1.9 c),g).

g) follows from f).

**COROLLARY 3.2.2** We use the notation of Proposition 3.2.1 and take $\psi := (α, β, γ, ε) ∈ A$.

a) Assume $ε = 1$ and there are $x, y ∈ Un E$ with $x^2 = βγ$, $y^2 = αγ$. Put $z := xyy^*$.

a1) $x, y, z ∈ Un Ec, z^2 = αβ$.

a2) For every $λ, μ ∈ \{-1, 1\}$ the map $φ_{λ, μ}$:

$$φ_{λ, μ} : S(f_0) \rightarrow E, \quad X \mapsto X_0 + λxX_a + μyX_b + λμzX_c$$

is an $E$-$C^*$-homomorphism.

a3) The map

$$S(f_0) \rightarrow E^4, \quad X \mapsto (φ_{1, 1}X, φ_{1, -1}X, φ_{-1, 1}X, φ_{-1, -1}X)$$

is an $E$-$C^*$-isomorphism.

b) Assume $K := \mathbb{K}, ε = 1$, and there are $x, y ∈ Un E$ with $x^2 = −βγ$, $y^2 = αγ$.

Put $z := xyy^*$. Then $x, y, z ∈ Un Ec, z^2 = −αβ$ (resp. $z^2 = αβ$), and the maps

$$S(f_0) \rightarrow \hat{E}, \quad X \mapsto (X_0 + ixX_a + yX_b + iX_aX_0 + X_aX_0)$$

$$S(f_0) \rightarrow \hat{E}, \quad X \mapsto (X_0 + ixX_a + yX_b - zX_c, X_0 + ixX_a - yX_b + zX_c)$$

are respectively $E$-$C^*$-isomorphisms (where $E$ denotes the complexification of $E$).

c) Assume $K := \mathbb{K}, ε = -1$, and there are $x, y ∈ E$ with $x^2 = −βγ$, $y^2 = αγ$. Put $z := xyy^*$. Then $x, y, z ∈ Un E^c, z^2 = −αβ$, and the map

$$S(f_0) \rightarrow \mathbb{K} ⊗ E, \quad X \mapsto X_0 + ixX_a + jyX_b + kzX_c,$$

where $i, j, k$ are the canonical units of $IH$, is an $E$-$C^*$-isomorphism.

d) If $ε = -1$ and there is an $x ∈ Un E$ with $x^2 = αβ$ then for every $δ ∈ Un E^c$ the map

$$S(f_0) \rightarrow E_{2, 2}, \quad X \mapsto \begin{bmatrix} X_0 + xX_a + yδ(βX_a - xX_b) \\ δ(X_a + xβX_b) \end{bmatrix}$$

is an $E$-$C^*$-isomorphism.

The proof is a long calculation using Proposition 3.2.1 f).

Remarks. d) is contained in Proposition 3.2.3 c). An example with $ε = 1$ but different from a) is presented in Proposition 3.3.2.
PROPOSITION 3.2.3 We use the notation of Proposition 3.2.1 and take \(\sigma := (\alpha, \beta, \gamma, \varepsilon) \in \mathcal{A}\).

(a) Let \(\varphi : S(f_\varepsilon) \to E_{2,2}\) be an \(E-C^*\)-isomorphism and put

\[
\begin{bmatrix}
A_t & B_t \\
C_t & D_t
\end{bmatrix} = \varphi V_t
\]

for every \(t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}\). Then \(\varepsilon = -1_E\), \(A_0, B_0, C_0, D_0 \in E^*\) and \(A_t + D_t = 0\) for every \(t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}\), and

\[
\begin{align*}
A^*_a &= \beta^\gamma A_a, & A^*_b &= -\alpha^\gamma A_b, & A^*_c &= \alpha^\beta A_c, \\
B^*_a &= \beta^\gamma A_a, & B^*_b &= -\alpha^\gamma C_b, & B^*_c &= \alpha^\beta C_c,
\end{align*}
\]

\[
A^2_a + B_a C_a = \beta y, \quad A^2_b + B_b C_b = -\alpha y, \quad A^2_c + B_c C_c = \alpha \beta,
\]

\[
A^2_a = \beta \gamma (1_E - |A_a|^2), \quad A^2_b = -\alpha \gamma (1_E - |B_b|^2), \quad A^2_c = \alpha \beta (1_E - |C_c|^2),
\]

\[
2A_a A_b + B_a C_b + B_b C_a = 0, \quad 2A_b A_c + B_b C_c + B_c C_b = 0,
\]

\[
2A_a A_b + B_a C_a + B_b C_b = 0,
\]

\[
\alpha A_a = A_a A_a + B_a C_a, \quad \alpha B_a = A_a B_a - A_b B_b, \quad \alpha C_a = A_a C_a - A_b C_b,
\]

\[
\beta A_b = A_a A_b + B_a C_b, \quad \beta B_b = A_a B_a - A_b B_b, \quad \beta C_b = A_a C_b - A_b C_b,
\]

\[
\gamma A_c = A_a A_c + B_a C_c, \quad \gamma B_c = A_a B_c - A_b B_c, \quad \gamma C_c = A_a C_c - A_b C_b,
\]

\[
|A_a| + |B_b| + |C_b| \neq 0, \quad |B_a| + |B_b| + |B_c| \neq 0, \quad |B_a| + |B_b| + |B_c| \neq 3.1_E.
\]

(b) Let \((A_a)_{\in \mathcal{D}}, (B_b)_{\in \mathcal{D}}, (C_c)_{\in \mathcal{D}}, (D_d)_{\in \mathcal{D}}\) be families in \(E^*\) satisfying the above conditions and put

\[
\begin{align*}
X' &= A_a X_a + A_b X_b + A_c X_c, & X' &= B_a X_a + B_b X_b + B_c X_c,
\end{align*}
\]

\[
X'' &= C_a X_a + C_b X_b + C_c X_c
\]

for every \(X \in S(f_\varepsilon)\). If \(\varepsilon = -1_E\) then the map

\[
S(f_\varepsilon) \to E_{2,2}, \quad X \mapsto \begin{bmatrix} X_0 + X' & X'' \\ X' & X_0 - X'' \end{bmatrix}
\]

is an \(E-C^*\)-isomorphism.

(c) Let \(\varepsilon = -1_E\) and assume there is an \(x \in E^*\) with \(x^2 = \beta \gamma\). Let \(y \in \mathcal{U} E^*\) and put \(z := \gamma^* x^y\). Then \(x, y, z \in \mathcal{U} E^*\) and the map

\[
\varphi : S(f_\varepsilon) \to E_{2,2}, \quad X \mapsto \begin{bmatrix} X_0 + x X_a & \alpha (y X_b + z X_c) \\ -y y^* X_b + \beta z^* X_c & X_0 - x X_a \end{bmatrix}
\]

is an \(-E-C^*\)-isomorphism such that

\[
\varphi(\frac{1}{2} (V_0 + (x^* \otimes 1_E)V_0)) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

In particular (by the symmetry of \(a, b, c\)), if \(\varepsilon = -1_E\) and if there is an \(x \in E^*\) with \(x^2 = \beta \gamma\), or \(x^2 = -\alpha^\gamma\), or \(x^2 = \alpha \beta\) then \(S(f_\varepsilon) \cong E_{2,2}\).

Remark: Take \(\sigma := (1_E, 1_E, -1_E, 1_E)\) and \(\sigma' := (1_E, 1_E, y^*, -1_E)\). By (c), \(S(f_\varepsilon) \cong E_{2,2}\) and by Proposition 3.2.1 \(c_1 \Rightarrow c_2\), \(S(f_\varepsilon) \cong S(f_\varepsilon')\) implies the existence of an \(x \in \mathcal{U} E^*\) with \(x^2 = y^*\).
COROLLARY 3.2.4 We use the notation of Proposition 3.2.3 and take \( E = \mathbb{K} \), \( \alpha = 1 \), and \( \beta = \gamma = \varepsilon = -1 \). Let \( S \) be a group, \( F \) a unital C*-algebra, \( g \in F(S, F) \), and
\[
h : ((S \times (\mathbb{Z}_2)^2) \times (S \times (\mathbb{Z}_2)^2)) \rightarrow, \quad \text{Un} \quad F^\prime, \quad ((s_1, t_1), (s_2, t_2)) \mapsto f(t_1, t_2)g(s_1, s_2).
\]

a) \( h \in F(S \times (\mathbb{Z}_2)^2, F) \),
b) \( S(h) \approx S(g)_{1,2} \), \( S_{1,2}(h) \approx S_{1,2}(g)_{1,2} \).

By Proposition 3.2.3 c), \( S(f) \approx \mathbb{K}_{2,2} \), so by Proposition 2.2.11 c), e),
\[
S(h) \approx \mathbb{K}_{2,2} \otimes S(g) = S(g)_{1,2}, \quad S_{1,2}(h) \approx \mathbb{K}_{2,2} \otimes S_{1,2}(g) \approx S_{1,2}(g)_{1,2}.
\]

EXAMPLE 3.2.5 Let \( \mathbb{K} = \mathbb{C} \) and \( E = \mathbb{C}( T, \mathbb{C}) \).

a) With the notation of Proposition 3.2.1, if \( \varrho = (\alpha, \beta, \gamma, -1) \in \mathbb{A} \) then \( S(f) \approx E_{2,2} \).
b) \( \text{Card} \{ (S(f) \mid f \in F(\mathbb{Z}_2 \times \mathbb{Z}_2, E) \} / \approx_S \} = 16 \).

Put
\[
m = w(\alpha), \quad n = w(\beta), \quad p = w(\gamma),
\]
where \( w \) denotes the winding number. By Proposition 2.2.2 \( a_1 \Rightarrow a_2 \), we may assume \( \alpha = z^m, \beta = z^n, \gamma = z^p \).

a) If \( n + p \) is even then the assertion follows from Proposition 3.2.3 c). If \( n + p \) is odd then either \( m + p \) or \( m + n \) is even and the assertion follows again from Proposition 3.2.3 c).
b) follows from Proposition 2.2.2 a), c).

Remark. Assume \( \mathbb{K} = \mathbb{R} \) and let \( E \) be the real C*-algebra \( \mathbb{C}( T, \mathbb{C}) \) ([C1] Theorem 4.1.1.8 a)), \( \varepsilon = -1_x \).
\[
\alpha : T \rightarrow \mathbb{C}, \quad z \mapsto z, \quad \beta : T \rightarrow \mathbb{C}, \quad z \mapsto -z, \quad \gamma : T \rightarrow \mathbb{C}, \quad z \mapsto e^z,
\]
and \( \varrho = (\alpha, \beta, \gamma, \varepsilon) \). Then by Corollary 3.2.2 c), \( S(f) \approx \mathbb{K} \otimes E \).

EXAMPLE 3.2.6 We put \( E = \mathbb{C}( T^2, \mathbb{C}), \gamma := 1_x \).
\[
\alpha : T^2 \rightarrow \mathbb{C}, \quad (z_1, z_2) \mapsto z_1, \quad \beta : T^2 \rightarrow \mathbb{C}, \quad (z_1, z_2) \mapsto z_2,
\]
and (with the notation of Proposition 3.2.1) \( \varrho = (\alpha, \beta, \gamma, -1_x) \in \mathbb{A} \).

a) \( S(f) \) is not commutative and not \( E \)-C*-isomorphic to \( E_{2,2} \).
b) If we put
\[
\tilde{x} : T^2 \rightarrow \mathbb{C}, \quad (z_1, z_2) \mapsto x(z_1^2, z_2^2)
\]
for every \( x \in E \) then the map
\[
S(f) \rightarrow E_{2,2}, \quad X \mapsto \begin{bmatrix} \tilde{x}_0 + \alpha \tilde{x}_c & \beta \tilde{x}_d - \alpha \tilde{x}_b \\ \beta \tilde{x}_a + \alpha \tilde{x}_b & \tilde{x}_0 - \alpha \beta \tilde{x}_c \end{bmatrix}
\]
is a C*-isomorphism.
c) \( E_{2,2} \approx S(f) \approx_E E_{2,2} \).
a) By Proposition 3.2.1 d), $S(f^e)$ is not commutative. Assume $S(f^e) \approx E_{2,2}$ and let us use the notation of Proposition 3.2.3 a).

Step 1 $\{A_0 \neq 0\} \subset \{A_b = 0\}$

Assume $\{A_a \neq 0\} \cap \{A_b \neq 0\} \neq \emptyset$. By Proposition 3.2.3 a),

$$2A_a A_b + B_a C_b + B_b C_a = 0,$$

$B'_a = \beta C_a$, $B'_b = \alpha C_b$

so $B_a \neq 0$ and $B_b \neq 0$ on this set. We put

$$A_a := |A_a| e^{i \theta_a}, A_b := |A_b| e^{i \theta_b}, A_c := |A_c| e^{i \theta_c}, \quad z_1 := e^{i \theta_a}, z_2 := e^{i \theta_b},$$

with $\tilde{A}_a, \tilde{A}_b, \tilde{B}_a, \tilde{B}_b \in \mathbb{R}$. By Proposition 3.2.3 a), $2 \tilde{A}_a = \theta_1, 2 \tilde{A}_b = \theta_1 + \pi$,

$$B_a C_b + B_b C_a = -\alpha \gamma B_a B'_a + \beta \gamma B_b B'_b = |B_a||B_b|(e^{i(\theta_1 + \theta_2 + \pi)} - e^{i(\theta_1 + \theta_2 + \pi)}) =$$

$$= |B_a||B_b|e^{i(\theta_1 + \theta_2 + \pi)}(e^{i(\theta_1 + \theta_2 + \pi)} - e^{i(\theta_1 + \theta_2 + \pi)}) = 2|B_a||B_b|\sin(\frac{\theta_1 - \theta_2}{2} + \tilde{B}_b - \tilde{B}_a)e^{i(\theta_1 + \theta_2 + \pi)}.$$

Since $2A_a A_b = -(B_a C_b + B_b C_a)$ there is a $k \in \mathbb{Z}$ with

$$\frac{\theta_1}{2} + \frac{\theta_2 + \pi}{2} = \frac{\theta_1 + \theta_2 + \pi}{2} + (2k + 1)\pi$$

which is a contradiction.

Step 2 $\{A_a \neq 0\} \subset \{A_0 = 0\}$

The assertion follows from Step 1 by symmetry.

Step 3 $\{A_a \neq 0\} = \{A_b = A_c = 0\}$

The assertion follows from Steps 1 and 2 and from $|A_a| + |A_b| + |A_c| \neq 0$.

Step 4 The contradiction

By Step 3 and by the symmetry, the sets $\{A_a \neq 0\}$, $\{A_b \neq 0\}$, and $\{A_c \neq 0\}$ are clopen and by $|A_a| + |A_b| + |A_c| \neq 0$ their union is equal to $\mathbb{T}^2$. So there is exactly one of these sets equal to $\mathbb{T}^2$ which implies

$$A^2_a = z_2, \quad A^2_b = -z_1, \quad A^2_c = z_1 z_2$$

and no one of these identities can hold.

b) is a direct verification.

c) follows from a) and b).

3.3 $T := (\mathbb{Z}/2)^n$ with $n \in \mathbb{N}$

**EXAMPLE 3.3.1** Assume $f$ constant and put

$$\langle s|t \rangle := \prod_{i=1}^{n} (-1)^{\langle s|t \rangle}$$

for all $s,t \in T$ (where $\mathbb{Z}/2$ is identified with $\{0,1\}$) and

$$\varphi_e : S(f) \to E, \quad X \to \sum_{s \in T} \langle t|s \rangle X_s$$
for all \( t \in T \). Then the map

\[
\varphi : S(f) \rightarrow E^n, \quad X \mapsto (\varphi_tX)_{t \in T}
\]

is an \( E\text{-C}^* \)-isomorphism.

For \( r, s, t \in T \),

\[
t + t = 0, \quad s(t) = (t)s, \quad (r + s)t) = (r)t(s)t, \quad (r)s + t = (r)s(r)t.
\]

For \( t \in T \) and \( X, Y \in S(f) \), by Theorem 2.1.9 c), g),

\[
(\varphi_tX)(\varphi_tY) = \sum_{r \in T} (t|r)(t)sXsY = \sum_{q \in T} (t|q)(q|r)XYq = \\
= \sum_{q \in T} (t|q)XY_q = \varphi_t(XY)
\]

so \( \varphi_t \) and \( \varphi \) are \( E\text{-C}^* \)-homomorphisms.

We have

\[
\sum_{t \in T} (0|t) = 2^n.
\]

We want to prove

\[
\sum_{t \in T} (s|t) = 0.
\]

for all \( s \in T, s \neq 0 \), by induction with respect to \( \text{Card}\{i \in \mathbb{N}_n \mid s(i) \neq 0\} \). Let \( i \in \mathbb{N}_n \) with \( s(i) \neq 0 \) and put \( r := s + e_i \)

Then

\[
\sum_{t \in T_0} (s|t) = \sum_{t \in T_1} (r|t), \quad \sum_{t \in T_0} (s|t) = -\sum_{t \in T_1} (r|t).
\]

But

\[
\sum_{t \in T_0} (r|t) = \sum_{t \in T_1} (r|t) = 2^{n-1}
\]

if \( r = 0 \). By the hypothesis of the induction

\[
\sum_{t \in T_0} (r|t) = \sum_{t \in T_1} (r|t) = 0
\]

if \( r \neq 0 \) (with \( \mathbb{N}_n \) replaced by \( \mathbb{N}_n \setminus \{i\} \), since \( r(i) = 0 \)). This finishes the proof by induction.

For \( r \in T \) and \( X \in S(f) \), by the above,

\[
\sum_{t \in T} (r|t)\varphi_tX = \sum_{s \in T} (r|t)(t|s)Xs = \sum_{s \in T} (r + s|t)Xs = \\
= \sum_{s \in T \setminus \{r\}} \sum_{t \in T} (r + s|t)Xs + \sum_{t \in T} (0|t)Xs = 2^nXr.
\]

Hence \( \varphi \) is bijective.
EXAMPLE 3.3.2 Let \( E \equiv \mathbb{C}(T^n, \mathbb{C}) \), denote by \( z := (z_1, z_2, \ldots, z_n) \) the points of \( T^n \), and put \( z^2 := (z_1^2, z_2^2, \ldots, z_n^2) \) for every \( z \in T^n \). We identify \( (\mathbb{Z}_2)^n \) with \( \Psi(\mathbb{Z}_n) \) by using the bijection

\[
\Psi(\mathbb{Z}_n) \to (\mathbb{Z}_2)^n, \quad I \mapsto e_I
\]

and denote by

\[
I \Delta := (I \setminus J) \cup (J \setminus I)
\]

the addition on \( \Psi(\mathbb{Z}_n) \) corresponding to this identification. We put \( \lambda_I := \prod_{i \in I} z_i \) for every \( I \subseteq \mathbb{Z}_n \) and

\[
f : \Psi(\mathbb{Z}_n) \times \Psi(\mathbb{Z}_n) \to \mathbb{C}^e, \quad (I, J) \mapsto \lambda_{I \setminus J}.
\]

Then \( f \in \mathcal{F}((\mathbb{Z}_2)^n, \mathbb{E}) \) and, if we put

\[
\overline{X} := \sum_{I \subseteq \mathbb{Z}_n} \lambda_I (x^2)_{\bar{I}} \in \mathbb{E}
\]

for every \( X \in \mathcal{S}(f) \), the map

\[
\varphi : \mathcal{S}(f) \to \mathbb{E}, \quad X \mapsto \overline{X}
\]

is an isomorphism of \( C^* \)-algebras.

Let \( X, Y \in \mathcal{S}(f) \). By Theorem 2.1.9 (c),

\[
\overline{X} = \sum_{I \subseteq \mathbb{Z}_n} \lambda_I (X^2)_I = \sum_{I \subseteq \mathbb{Z}_n} \lambda_I \overline{X}_I = \overline{X},
\]

\[
\overline{X} \overline{Y} = \sum_{I \subseteq \mathbb{Z}_n} \lambda_I (XY)_I = \sum_{I \subseteq \mathbb{Z}_n} \lambda_I \sum_{J \subseteq \mathbb{Z}_n} f(J, J \Delta)^2 Y_{I \Delta} X_{I \setminus J \Delta} = \sum_{J \subseteq \mathbb{Z}_n} \lambda_J Y_{J \setminus K} X_{K} = \overline{X} \overline{Y}
\]

so \( \varphi \) is a \( C^* \)-homomorphism.

We put for \( k \in \mathbb{Z}_n \), \( i \in \mathbb{Z}^n \), and \( I \subseteq \mathbb{Z}_n \),

\[
\beta^k_i := \begin{cases} 2k + 1 & \text{if } k \in I, \\ 2k & \text{if } k \in \mathbb{Z}_n \setminus I \end{cases}, \quad i^\ast := (i^1, i^2, \ldots, i^n) \in \mathbb{Z}^n
\]

and

\[
\mathcal{G} := \left\{ \sum_{i \in \mathbb{Z}^n} a_i x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid (a_i)_{i \in \mathbb{Z}^n} \in \mathbb{C}^{(n)} \right\}.
\]

Let

\[
x = \sum_{i \in \mathbb{Z}^n} a_i x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in \mathcal{G}
\]

and for every \( I \subseteq \mathbb{Z}_n \) put

\[
X_I := \sum_{i \in \mathbb{Z}_n} a_i x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in \mathcal{G}, \quad X := \sum_{I \subseteq \mathbb{Z}_n} (X_I \otimes 1_K) V_I.
\]

Then \( \varphi X = x \) and so \( \mathcal{G} \subseteq \varphi(\mathcal{S}(f)) \). Since \( \mathcal{G} \) is dense in \( \mathbb{C}^e \), it follows that \( \varphi \) is surjective.

We prove that \( \varphi \) is injective by induction with respect to \( n \in \mathbb{N} \). The case \( n = 1 \) was proved in Example 3.1.4. Assume the assertion holds for \( n - 1 \). Let \( X \in \text{Ker } \varphi \). Then

\[
\sum_{I \subseteq \mathbb{Z}_n} \lambda_I (x^2) = 0.
\]
By replacing $z_n$ by $-z_n$ in the above relation, we get

$$\sum_{j \in \mathbb{N}_n} \lambda_j(z)X_j(z^2) - \sum_{n \in \mathbb{N}_n} \lambda_j(z)X_j(z^2) = 0$$

and so

$$\sum_{j \in \mathbb{N}_n} \lambda_j(z)X_j(z^2) = \sum_{n \in \mathbb{N}_n} \lambda_j(z)X_j(z^2) = 0.$$

By the induction hypothesis, we get $X_i = 0$ for all $i \in \mathbb{N}_n$ and so $X = 0$. Thus $\varphi$ is injective and a $C^*$-isomorphism.

**EXAMPLE 3.3.3** Let $f \in \mathcal{F}(\mathbb{Z}_2^3, E)$, put

$$a = (0, 0, 1), \quad b = (0, 1, 0), \quad c = (0, 1, 1), \quad s = (1, 0, 0),$$

and denote by $g$ the element of $\mathcal{F}(\mathbb{Z}_2, E)$ defined by $g(1, 1) = f(s, s)$ Proposition 3.1.1 a).

a) There is a family $(\alpha, \beta, \gamma, \epsilon)_{i \in \mathbb{N}_7}$ in $(\text{Un} E^3)^4$ such that $f$ is given by the attached table and such that $\sum_{i \in \mathbb{N}_7} \epsilon_i^2 = 1_{E}$ for every $i \in \mathbb{N}_7$ and

$$\begin{align*}
\epsilon_3 &= \epsilon_1 \epsilon_2, \quad \epsilon_5 = \epsilon_1 \epsilon_4, \quad \epsilon_6 = \epsilon_2 \epsilon_4, \quad \epsilon_7 = \epsilon_1 \epsilon_2 \epsilon_4, \\
\alpha_3 &= \epsilon_2 \epsilon_4 \alpha_3 \epsilon_2 \alpha_4 \alpha_6 \gamma_2, \quad \alpha_5 = \epsilon_3 \epsilon_2 \beta_1 \gamma_2, \quad \alpha_7 = \alpha_1 \gamma_1 \gamma_2, \\
\beta_2 &= \epsilon_1 \gamma_1 \gamma_2, \quad \beta_3 = \epsilon_2 \gamma_3 \gamma_2, \quad \beta_4 = \epsilon_1 \epsilon_2 \epsilon_4 \alpha_2 \gamma_1 \gamma_2, \\
\beta_5 &= \epsilon_1 \epsilon_2 \gamma_2, \quad \beta_6 = \epsilon_1 \epsilon_2 \gamma_1 \gamma_2, \quad \beta_7 = \epsilon_1 \epsilon_2 \epsilon_4 \alpha_2 \gamma_2, \\
\gamma_3 &= \epsilon_2 \epsilon_4 \alpha_2 \gamma_1, \quad \gamma_4 = \epsilon_2 \epsilon_4 \alpha_3 \gamma_1, \quad \gamma_5 = \epsilon_1 \epsilon_2 \epsilon_4 \gamma_1, \\
\gamma_6 &= \epsilon_4 \epsilon_2 \alpha_2 \gamma_2, \quad \gamma_7 = \epsilon_1 \epsilon_2 \epsilon_4 \alpha_2 \beta_1.
\end{align*}$$

b) If $\sum_{i \in \mathbb{N}_7} \epsilon_i^2 = 1_{E}$, and there is an $x \in E$ with $x^2 = \alpha_1 \beta_1 \beta_1$ then there are $P_+ \in (E \otimes 1_{K^1})^I \cap \text{Pr}S(f)$ with $P_+ + P_- = V_4^2$ and (Theorem 2.2.18 b))

$$P_+ S(f) P_+ \cong_{E} S(g) \cong_{E} P_- S(f) P_-.$$

c) If $\sum_{i \in \mathbb{N}_7} \epsilon_i^2 = 1_{E}$, and there is an $x \in E$ with $x^2 = \alpha_1 \beta_1 \beta_1$ then $S(f) \cong_{E} S(g) 2_2$.  

d) Assume $\epsilon_1 = -1_{E}, \epsilon_2 = \epsilon_4 = \epsilon_5 = 1_{E}$ and there is an $x \in E$ with $x^2 = \alpha_1 \beta_1 \beta_1$ then $S(f) \cong_{E} 2_{2}.  

\text{Assume } \epsilon_1 = -1_{E}, \epsilon_2 = \epsilon_4 = \alpha_1 = \beta_1 = \gamma_1 = 1_{E}, \gamma_2 = \alpha_2, \text{ and } \alpha_2^2 = \alpha_4 = \alpha_6 = 1_{E} \text{ and put }\n
\varphi_{\pm} : S(f) \rightarrow E_{2}, \quad X \mapsto
\begin{bmatrix}
X_0 + X_1 \pm X_2 \pm X_3 \pm X_4 \\ X_5 - X_6 \pm X_7 \pm X_8 \pm X_9 \pm X_{10} \pm X_{11} \pm X_{12} \pm X_{13} \\
X_6 - X_5 \pm X_8 \pm X_9 \pm X_{10} \pm X_{11} \pm X_{12} \pm X_{13}
\end{bmatrix}\text{.} }
Then the map

\[ S(f) \to E_{2,2} \times E_{2,2}, \quad X \mapsto (\varphi, X, \varphi, X) \]

is an E-C*-isomorphism.

a) is a long calculation.
b) and c) follow from a) and Theorem 2.2.18 e).
d) is a long calculation using a).

\[ \square \]

### 3.4 \( T := \mathbb{Z}_n \) with \( n \in \mathbb{N} \)

**Proposition 3.4.1** Put \( A := \text{Un} \ E \) and for every \( \alpha \in A^{n-1} \) put

\[ f_\alpha : \mathbb{Z}_n \times \mathbb{Z}_n \to A, \quad (p, q) \mapsto (\prod_{j=p}^{p+q-1} \alpha_j)(\prod_{k=1}^{q-1} \alpha_k). \]

where \( \mathbb{Z}_n \) and \( \mathbb{N}_n \) are canonically identified and \( \alpha_n := 1_E \).

a) For every \( f \in \mathcal{F}(\mathbb{Z}_n, E) \) and \( X \in S(f) \), \( X \in S(f) \) iff \( X_t \in Ec \) for all \( t \in T \). In particular, \( S(f) \) is commutative if \( E \) is commutative.

b) \( f_\alpha \in \mathcal{F}(\mathbb{Z}_n, E) \) for every \( \alpha \in A^{n-1} \) and the map

\[ A^{n-1} \to \mathcal{F}(\mathbb{Z}_n, E), \quad \alpha \mapsto f_\alpha \]

is a group isomorphism.

c) The following are equivalent for all \( \alpha, \beta \in A^{n-1} \).

\[ c_1 \] \( S(f_\alpha) \cong S(f_\beta) \).

\[ c_2 \] There is a \( \gamma \in A \) such that

\[ \gamma^n = \prod_{j=1}^{n-1} (\alpha_j \beta_j). \]

c) There is a \( \lambda \in \Lambda(\mathbb{Z}_n, E) \) such that \( f_\alpha \circ f_\beta = f_\lambda \).

If these equivalent conditions are fulfilled then the map

\[ S(f_\alpha) \to S(f_\beta), \quad X \mapsto U_\alpha X U_\beta \]

is an \( S \) -isomorphism and

\[ \lambda(1)^n = \prod_{j=1}^{n-1} (\alpha_j \beta_j) = \gamma^n, \quad p \in \mathbb{Z}_n \Rightarrow \lambda(p) = \lambda(1)^p \prod_{j=1}^{n-1} (\alpha_j \beta_j). \]

d) Let \( \alpha \in A^{n-1} \) and put

\[ \beta : \mathbb{N}_{n-1} \to A, \quad j \mapsto \left\{ \begin{array}{ll} 1 & \text{if } j < n-1 \\ (\prod_{k=1}^{n-1} \alpha_k)^{n-1} & \text{if } j = n-1. \end{array} \right. \]

Then \( \alpha \) and \( \beta \) fulfill the equivalent conditions of c).

e) There is a natural bijection

\[ \{ S(f) \mid f \in \mathcal{F}(\mathbb{Z}_n, E) \} / \approx_S \to \mathbb{A} / \{ x^n \mid x \in A \} \]

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If \( E = \mathcal{C}(\mathbb{T}^m, \mathbb{C}) \) for some \( m \in \mathbb{N} \) then

\[
\text{Card}(\{f \mid f \in \mathcal{F}(\mathbb{Z}_n, E)\}/\sim_S) = mn.
\]

f) Let \( \alpha \in A^{n-1}, \beta \in A \) such that \( \beta^* = \prod_{j=1}^{n-1} \alpha_j \),

\[ E := \begin{cases} 
\mathbb{C} & \mathbb{K} = \mathbb{C} \\
\mathbb{E} & \mathbb{K} = \mathbb{R}
\end{cases} \]

where \( \mathbb{E} \) denotes the complexification of \( E \), and

\[ w_k : S(f) \to \mathbb{E}, \quad X \mapsto \sum_{j=1}^{n} \beta^* \left( \prod_{j=1}^{n-1} \alpha_j \right)^{2\pi i X_j}
\]

for every \( k \in \mathbb{N}_n(= \mathbb{Z}_n) \).

fi) If \( \mathbb{K} = \mathbb{C} \) then the map

\[ S(f) \to \mathbb{E}, \quad X \mapsto (w_k X)_{k \in \mathbb{Z}_n}
\]

is an \( E \)-C*-isomorphism.

fii) If \( \mathbb{K} = \mathbb{R} \) and \( n \) is odd then we may take \( \beta \in \mathbb{R} \) and the map

\[ S(f) \to \mathbb{E} \times (\mathbb{E})^\mathbb{Z}, \quad X \mapsto (w_n X, (w_k X)_{k \in \mathbb{N}_n})
\]

is an \( E \)-C*-isomorphism.

fiii) If \( \mathbb{K} = \mathbb{R} \), \( n \) is even, and \( \prod_{j=1}^{n-1} \alpha_j = -1 \) then the map

\[ S(f) \to (\mathbb{E})^{2}, \quad X \mapsto (w_{k-1} X)_{k \in \mathbb{N}_n}
\]

is an \( E \)-C*-isomorphism.

fiv) If \( \mathbb{K} = \mathbb{R} \), \( n \) is even, and \( \prod_{j=1}^{n-1} \alpha_j = 1 \), and \( \beta = 1 \) then the map

\[ S(f) \to \mathbb{E} \times (\mathbb{E})^{\mathbb{Z}}, \quad X \mapsto (w_n X, w_k X, (w_k X)_{k \in \mathbb{N}_n})
\]

is an \( E \)-C*-isomorphism.

fvi) If \( n \) is even then there is a \( \gamma \in A \) such that \( f_\alpha(\frac{1}{2}, \frac{1}{2}) = \gamma^2 \).

\[ \Box \]

**EXAMPLE 3.4.2** Let \( E = \mathcal{C}(\mathbb{T}, \mathbb{C}), r \in \mathbb{Z}^{n-1}, z : \mathbb{T} \to \mathbb{C} \) the canonical inclusion, and

\[ f : \mathbb{Z}_n \times \mathbb{Z}_n \to \text{Un} E^c, \quad (p, q) \mapsto r \left( \sum_{j=p}^{q-1} r_j - \sum_{j=1}^{q-1} r_j \right), \]

where \( \mathbb{Z}_n \) and \( \mathbb{N}_n \) are canonically identified. Then \( f \in \mathcal{F}(\mathbb{Z}_n, E) \). Let further \( S \) be the subgroup of \( \mathbb{Z}_n \) generated by \( \rho \left( \sum_{j=1}^{n-1} r_j \right) \), where \( \rho : \mathbb{Z} \to \mathbb{Z}_n \) is the quotient map,

\[ m = \text{Card } S, \quad h = \frac{n}{m}, \quad \omega = e^{2\pi i}, \]

\[ \sigma : \mathbb{N}_n \to \mathbb{Z}, \quad p \mapsto \frac{p n-1}{\sum_{j=1}^{n-1} r_j}, \]

and

\[ \varphi_k : S(f) \to \mathbb{E}, \quad X \mapsto \sum_{p=1}^{n} (X_p \omega^{m p})^2 \omega^{m k} \]
for every \( k \in \mathbb{N}_n \). Then the map

\[
\phi : S(f) \rightarrow E^n, \quad X \mapsto (\phi_k X)_{k \in \mathbb{N}_n}
\]

is an \( E\)-\( C^* \)-isomorphism.

The next example shows that the set \( \{ S(f) \mid f \in \mathcal{F}(\mathbb{Z}_n, \mathcal{C}(\mathbb{T}, \mathbb{C})) \} \) is not reduced by restricting the Schur functions to have the form indicated in Example 3.4.2.

**EXAMPLE 3.4.3** Let \( E = \mathcal{C}(\mathbb{T}, \mathbb{C}) \) and \( g \in \mathcal{F}(\mathbb{Z}_n, E) \). Put

\[
\varphi : [0, 2\pi] \rightarrow \mathbb{R}, \quad \theta \mapsto \log \prod_{j=1}^{n-1} (g(j, 1))(e^{i\theta}),
\]

where we take a fixed (but arbitrary) branch of \( \log \). If we define

\[
r : \mathbb{N}_{n-1} \rightarrow \mathbb{Z}, \quad j \mapsto \begin{cases} 
\lim_{\theta \rightarrow 2\pi} \varphi(\theta) - \varphi(0) & \text{if } j = 1 \\
0 & \text{if } j \neq 1
\end{cases}
\]

then there is a \( \lambda \in \Lambda(\mathbb{Z}_n, E) \) such that \( g = f\delta\lambda \), where \( f \) is the Schur function defined in Example 3.4.2. In particular \( S(f) \cong S(g) \).

**3.5** \( T = \mathbb{Z} \)

**EXAMPLE 3.5.1** Let \( f \in \mathcal{F}(\mathbb{Z}, E) \).

a) \( S_{\|f\|}(f) \cong \mathcal{C}(T, E) \).

b) If \( E \) is a \( W^* \)-algebra then

\[
S_W(f) \cong E \otimes L^n(\mu) \cong L^n(\mu, E),
\]

where \( \mu \) denotes the Lebesgue measure on \( T \).

By Corollary 1.1.6 c) and Proposition 2.2.2 a) \( a_1 \Rightarrow a_2 \), we may assume \( f \) constant. By Proposition 2.2.10 c), we may assume \( E = \mathbb{C} \). Let \( \alpha : T \rightarrow \mathbb{C} \) be the inclusion map. Then

\[
L^2(\mathbb{Z}) \rightarrow L^2(\mu), \quad \xi \mapsto \sum_{n \in \mathbb{Z}} \xi_n \alpha^n
\]

is an isomorphism of Hilbert spaces. If we identify these Hilbert spaces using this isomorphism then \( V_1 \) becomes the multiplicator operator

\[
L^2(\mu) \rightarrow L^2(\mu), \quad \eta \mapsto a\eta
\]

so

\[
R(f) \rightarrow L^\omega(\mu), \quad X \mapsto \sum_{n \in \mathbb{Z}} X_n \alpha^n
\]

is an injective, involutive algebra homomorphism. The assertion follows.
Throughout this section \( I \) is a totally ordered set, \((T_i)_{i \in I}\) is a family of groups, and \((f_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(T_i, E)\). We put
\[
T := \{ i \in I \mid t_i \neq 1 \}
\]
for every \( t \in \prod_{i \in I} T_i \) (where 1, denotes the neutral element of \( T_i \)) and
\[
T := \left\{ t \in \prod_{i \in I} T_i \mid T \text{ is finite} \right\},
\]
\[
T' := \{ t \in T \mid t^2 = 1 \}.
\]

An associated \( f \in \mathcal{F}(T, E) \) will be defined in Proposition 4.1.1 b).

\( T \) is a subgroup of \( \prod_{i \in I} T_i \). We canonically associate to every element \( t \in T \) in a bijective way the "word" \( t_{i_1} t_{i_2} \cdots t_{i_n} \) where \( \{i_1, i_2, \ldots, i_n \} = I \) and \( i_1 < i_2 < \cdots < i_n \)
and use sometimes this representation instead of \( t \) (to \( 1 \in T \) we associate the "empty word").

**PROPOSITION 4.1.1**

a) Let \( t_{i_1} t_{i_2} \cdots t_{i_k} \) be a finite sequence of letters with \( t_i \in T_i \setminus \{1\} \) for every \( j \in \mathbb{N}_n \) and use transpositions of successive letters with distinct indices in order to bring these indices in an increasing order. If \( \tau \) denotes the number of used transpositions then \((-1)^\tau \) does not depend on the manner in which this operation was done.

b) Let \( s, t \in T \) and let
\[
s_{i_1} s_{i_2} \cdots s_{i_n}, \quad t_{i_1} t_{i_2} \cdots t_{i_n}
\]
be the canonically associated words of \( s \) and \( t \), respectively. We put for every \( k \in I \), \( \overline{s}_k := s_k \) if there is a \( j \in \mathbb{N}_m \) with \( k = i_j \) and \( \overline{s}_k := 1_k \) if the above condition is not fulfilled and define \( \overline{t} \) in a similar way. Moreover we put (Proposition 1.1.2 a))
\[
f(s, t) := (-1)^\tau \prod_{k \in I} f_k(\overline{s}_k, \overline{t}_k),
\]
where \( \tau \) denotes the number of transpositions of successive letters with distinct indices in the finite sequence of letters
\[
s_{i_1} s_{i_2} \cdots s_{i_n}, t_{i_1} t_{i_2} \cdots t_{i_n}
\]
in order to bring the indices in an increasing order. Then \( f \in \mathcal{F}(T, E) \).

c) Let \( I_0 \) be a subset of \( I \), \( T_0 \) the subgroup \( \{ t \in T \mid t \subseteq I_0 \} \) of \( T \), and \( f_0 \) the element of \( \mathcal{F}(T_0, E) \) defined in a similar way as \( f \) was defined in b). Then \( f_0 = f \upharpoonright (T_0 \times T_0) \) and the map
\[
\mathcal{S} \upharpoonright (f_0) \mapsto \mathcal{S} \upharpoonright (f), \quad \sum_{t \in T_0} (X_t)_{k} V_{t}^{f_0} \mapsto \sum_{t \in T_0} (X_t)_{k} V_{t}^{f}
\]
is an injective \( E \cdot C^* \)-homomorphism with image
\[
\{X \in \mathcal{S}(f) \mid (t \in T) \neq 0 \Rightarrow t \in T_0 \}.
\]

a) We define a new total order relation on the indices of the given word by putting for all \( j, k \in \mathbb{N}_n \)
\[
i_j < i_k : \Leftrightarrow ((i_j < i_k) \text{ or } (i_j = i_k \text{ and } j < k)).
\]

Let \( P \) be a sequence of transpositions of successive letters in order to bring the indices in an increasing form with respect to the new order and let \( \tau' \) be the number of used transpositions. Then \( \tau - \tau' \) is even and so \((-1)^\tau = (-1)^\tau' \). By the theory of permutations
(-1)^ν does not depend on P, which proves the assertion.

b) By a), f is well-defined. Let r, s, t ∈ T and let

\[ r, r_2, \ldots, r_n, \quad s, s_2, \ldots, s_r, \quad t, t_2, \ldots, t_r, \]

be the words canonically associated to r, s, and t, respectively. There are α, β ∈ (-1, +1) such that

\[
f(r, s)f(rs, t) = \alpha \prod_{i \in I} f(r_i, s_i) f(r_i s_i, t_i),
\]

where

\[
f(r, st)f(s, t) = \beta \prod_{i \in I} f(r_i, s_i t_i) f(s_i, t_i).
\]

Write the finite sequence of letters

\[ r, r_2, \ldots, r_n, s, s_2, \ldots, s_r, t, t_2, \ldots, t_r, \]

and use transpositions of successive letters with distinct indices in order to bring the indices in an increasing order. We can do this acting first on the letters of \( r \) and \( s \) only and then in a second step also on the letters of \( t \). Then \( \alpha = (-1)^ν \), where \( ν \) denotes the number of all performed transpositions. For \( β \) we may start first with the letters of \( s \) and \( t \) and then in a second step also with the letters of \( r \). Then \( β = (-1)^ρ \), where \( ρ \) is the number of all effectuated transpositions. By a), \( \alpha = (-1)^ν \). The rest of the proof is obvious.

c) follows from Corollary 2.1.17 d).

**Corollary 4.1.2.** If \( I = \mathbb{N}_2 \) then for all s, t ∈ T,

\[
f(s, t) = \begin{cases} f_1(s_1, t_1) & \text{if} \quad s_2 = 1_2, \\ f_2(s_2, t_2) & \text{if} \quad t_1 = 1_1, \\ -f_1(s_1, t_1) f_2(s_2, t_2) & \text{if} \quad s_2 \neq 1_2, t_1 = 1_1. \end{cases}
\]

**Proposition 4.1.3.** Let s, t ∈ T.

a) \( f(s, t) = (-1)^{\text{Card}(s \times t) - \text{Card}(s \times t)} f(t, s) \).

b) \( st = ts \) if \( V_s V_t = (-1)^{\text{Card}(s \times t) - \text{Card}(t \times s)} V_t V_s \).

c) Assume \( s \subseteq t \). If \( \text{Card} s \) is even or if \( \text{Card} t \) is odd then \( f(s, t) = f(t, s) \). If in addition \( st = ts \) then \( V_s V_t = V_t V_s \).

d) If \( \text{Card} I \) is an odd natural number and \( T \) is commutative then \( V_t \in S(f)^ν \) for every \( t \in T \) with \( I = I \).

e) Assume \( t \in T' \). If \( n = \text{Card} \ T \) and \( \alpha = \prod_{i \in \mathcal{I}} f_i(t_i, t_i) \) then

\[
f(t, t) = (-1)^{\frac{n(n-1)}{2}} \alpha, \quad \bar{f}(t) = (-1)^{\frac{n(n-1)}{2}} \alpha^∗, \quad (V_t)^2 = (-1)^{\frac{n(n-1)}{2}} (\alpha \otimes 1_k) V_t, \quad V_t^* = (-1)^{\frac{n(n-1)}{2}} (\alpha^∗ \otimes 1_k) V_t.
\]

a) For \( i \in \mathcal{I} \),

\[
f(s_i, t) = \begin{cases} (-1)^{\text{card} \ T} f(t, s_i) & \text{if} \quad k \notin \mathcal{I}, \\ (-1)^{\text{card} \ T - 1} f(t, s_i) & \text{if} \quad i \in \mathcal{I}. \end{cases}
\]
so
\[
f(s, t) = (-1)^{\text{Card}(T \setminus T) - \text{Card}(\mathbb{T})} f(t, s).
\]

b) By Proposition 2.1.2 b),
\[
V_s V_t = (f(s, t) \otimes 1_K) V_{st}, \quad V_t V_s = (f(t, s) \otimes 1_K) V_{ts}.
\]

Thus if \( st = ts \) then by a),
\[
V_s V_t = ( (f(s, t)f(t, s)^*) \otimes 1_K ) V_{st} = (-1)^{\text{Card}(T \setminus T) - \text{Card}(\mathbb{T})} V_{ts}.
\]

Conversely, if this relation holds then by a),
\[
V_{st} = (f(s, t)^* \otimes 1_K) V_s V_t = (-1)^{\text{Card}(T \setminus T) - \text{Card}(\mathbb{T})} V_{ts}.
\]

and we get \( st = ts \) by Theorem 2.1.9 a).

c) follows from a) and b).

d) follows from c) (and Proposition 2.1.2 d)).

e) We have
\[
f(t, t) = (-1)^{n-1} \alpha^2 = (-1)^{\frac{n-2}{2}} \alpha.
\]

By Proposition 2.1.2 b), e),
\[
(V_t)^2 = (f(t, t)^* \otimes 1_K) V_t = (-1)^{\frac{n-2}{2}} (\alpha^2 \otimes 1_K) V_t,
\]
\[
V_t^* = f(t) V_{t+1} = f(t, t)^* V_t = (-1)^{\frac{n-2}{2}} (\alpha^2 \otimes 1_K) V_t.
\]

\[\blacksquare\]

**PROPOSITION 4.1.4.** Let \( S \) be a finite subset of \( T \setminus \{1\} \) such that \( st = ts \) and \( \text{Card} (S \times T) \setminus \text{Card}(T \setminus T) \) is odd for all distinct \( s, t \in S \) and for every \( t \in S \) let \( \alpha_t, \epsilon_t \in \mathbb{E}^\epsilon \) and \( X_t \in E \) be such that
\[
\epsilon_t^2 = 1_E, \quad (V_t)^2 = (\alpha_t^2 \otimes 1_K) V_t, \quad X_t^* = \alpha_t X_t,
\]
\[
\sum_{t \in S} |X_t|^2 = \frac{1}{4} 1_E.
\]

a)
\[
P = \frac{1}{4} V_1 + \sum_{t \in S} (\epsilon_t X_t^* \otimes 1_K) V_t \in \text{Pr } S(f),
\]
\[
V_1 - P = \frac{1}{2} V_1 + \sum_{t \in S} (-\epsilon_t X_t^* \otimes 1_K) V_t \in \text{Pr } S(f).
\]

b) If \( s \in S \) and \( \beta \in \mathbb{E}^\epsilon \) such that \( X_s = 0 \) and \( \beta^2 = \alpha_s \) then \( P \) is homotopic in \( \text{Pr } S(f) \) to
\[
\frac{1}{2} (V_1 + (\beta^* \epsilon_s \otimes 1_K) V_s).
\]

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a) By Proposition 4.1.3 b), e),

\[ P^* = \frac{1}{2} V_1 + \sum_{t \in S} \left( (\varepsilon_t^* X_t^* \alpha_t^*) \otimes 1_K \right) V_t = \frac{1}{2} V_1 + \sum_{t \in S} \left( (\varepsilon_t X_t) \otimes 1_K \right) V_t = P, \]

\[ P^2 = \frac{1}{4} V_1 + \sum_{t \in S} \left( (X_t^2) \otimes 1_K \right) V_t + \sum_{t \in S} \left( (\varepsilon_t X_t) \otimes 1_K \right) V_t + \sum_{s, t \in S, s \neq t} \left( (\varepsilon_s X_s) \otimes 1_K \right) (V_s V_t + V_t V_s) = \]

\[ = \frac{1}{4} V_1 + \sum_{t \in S} \left( (X_t^2) \otimes 1_K \right) V_t + \sum_{t \in S} \left( (\varepsilon_t X_t) \otimes 1_K \right) V_t = \]

\[ = \frac{1}{4} V_1 + \sum_{t \in S} \left( (X_t^2) \otimes 1_K \right) V_t + \sum_{t \in S} \left( (\varepsilon_t X_t) \otimes 1_K \right) V_t = P. \]

b) Remark first that \( \beta \in Un E^c \) and put

\[ Y : [0, 1] \to E_+^c, \quad u \mapsto \left( \frac{1}{4} e^{-u^2} \sum_{t \in S} |X_t|^2 \right)^{1/2}, \]

\[ Z : [0, 1] \to E^c, \quad u \mapsto \beta^* \varepsilon_s Y(u), \]

\[ Q : [0, 1] \to S(f), \quad u \mapsto \frac{1}{2} V_1 + (Z(u) \otimes 1_K) V_t + \sum_{t \in S \setminus \{s\}} \left( (\varepsilon_t X_t) \otimes 1_K \right) V_t. \]

For \( u \in [0, 1] \),

\[ \alpha_s Z(u) = \beta^2 \beta^* \varepsilon_s Y(u) = \beta \varepsilon_s Y(u) = Z(u)^*, \]

so by a), \( Q(u) \in \text{Pr} S(f) \). Moreover

\[ Q(0) = \frac{1}{2} (V_1 + ((\beta^* \varepsilon_s) \otimes 1_K) V_t), \quad Q(1) = P. \]

**COROLLARY 4.1.5.** Let \( s, t \in T \setminus \{1\}, s \neq t, st = ts, \alpha_s \neq \varepsilon_s \), \( \varepsilon_t \in Un E^c \) such that

\[ \varepsilon_s^2 = \varepsilon_t^2 = 1_K, \quad (V_s)^2 = (\alpha_s^2 \otimes 1_K) V_t, \quad (V_t)^2 = (\alpha_t^2 \otimes 1_K) V_t, \]

and put

\[ P_s := \frac{1}{2} (V_1 + ((\varepsilon_s \alpha_s^*) \otimes 1_K) V_t), \quad P_t := \frac{1}{2} (V_1 + ((\varepsilon_t \alpha_t^*) \otimes 1_K) V_t). \]

a) \( P_s, P_t \in \text{Pr} S(f) \); we denote by \( P_s \wedge P_t \) the infimum of \( P_s \) and \( P_t \) in \( S(f)_+ \) (by b) and c) it exists).

b) If \( V_s V_t \neq V_t V_s \) then \( P_s \wedge P_t = 0 \).

c) If \( V_s V_t = V_t V_s \) then \( P_s \wedge P_t = P_s P_t \in \text{Pr} S(f) \).

a) follows from Proposition 2.1.20 b. a.
b) By Proposition 4.1.3 b), $V_i V_j = -V_j V_i$. Let $X \in S(f)_+$ with $X \leq P_\omega$ and $X \leq P_\omega$. By [C1] Proposition 4.2.7.1 $d \Rightarrow c,$
\[
X = P_\omega X = \frac{1}{2} X + \frac{1}{2} \left( (\varepsilon, \alpha_1^* \otimes 1_k) V_\omega X, \right.
\]
\[
X = \left( (\varepsilon, \alpha_1^* \otimes 1_k) V_\omega X, \right. \]
\[
X = \left( (\varepsilon, \alpha_1^* \otimes 1_k) V_\omega X, \right. \]
\[
X = \left( (\varepsilon, \alpha_1^* \otimes 1_k) V_\omega X, \right. \]
\[
X = \left( (\varepsilon, \alpha_1^* \otimes 1_k) V_\omega X, \right. \]
so $X = 0$ and $P_\omega \land P_\omega = 0$.

We put $P_\omega P_\omega = P_\omega P_\omega$ so $P_\omega P_\omega \in \text{PrS}(f)$ and $P_\omega P_\omega = P_\omega \land P_\omega$ by [C1] Corollary 4.2.7.4 $a \Rightarrow b$.

**COROLLARY 4.1.6.** Let $m, n \in \mathbb{N}, n \in \mathbb{N}^+ - \mathbb{N}$, and for every $i \in \mathbb{N}$, let $t_i \in T$ with $T_i := \mathbb{N} \cup (n + 1)$ and $t_i t_j = t_j t_i$ for all $i, j \in \mathbb{N}_m$. If for every $i \in \mathbb{N}_m,$
\[
(V_i)^2 = (\alpha_i^* \otimes 1_k)V_i
\]
then
\[
\frac{1}{2} \left( V_i + \frac{1}{\sqrt{m}} \sum_{j \in \mathbb{N}_m} (\alpha_j^* \otimes 1_k)V_j \right) \in \text{PrS}(f).
\]

For distinct $i, j \in \mathbb{N}_m,
\[
\text{Card } (T_i \times T_j) \text{Card } (T_i \land T_j) = (n + 1)^2 - n = n(n + 1) + 1
\]
is odd. For every $i \in \mathbb{N}_m$ put $X_i := \frac{1}{\sqrt{m}} \alpha_i^*$. Then
\[
\alpha_i^2 X_i = \frac{1}{2 \sqrt{m}} \alpha_i = X_i, \quad |X_i|^2 = \frac{1}{4m} 1_E, \quad \sum_{i \in \mathbb{N}_m} |X_i|^2 = \frac{1}{4} 1_E
\]
and the assertion follows from Proposition 4.1.4 a).

**THEOREM 4.1.7.** Let $n \in \mathbb{N}$ such that $\mathbb{N}_{2n}$ is an ordered subset of $I$, $S := \{t \in T \mid \mathbb{N}_{2n-2} \subseteq T \}$, $g := f(S \times S)$, $a, b \in T$ such that $a^2 = b^2 = 1$,
\[
\mathbb{N} = \mathbb{N}_{2n-1}, \quad \mathbb{N} = \mathbb{N}_{2n-2} \cup \{2n\}, \quad i \in \mathbb{N}_{2n-2} \Rightarrow a_i = b_i.
\]
\[
\omega : \mathbb{Z} \times \mathbb{Z} \rightarrow T \text{ the (injective) group homomorphism defined by } \omega(1, 0) := a, \omega(0, 1) := b, \alpha_1 := f(a, a), \alpha_2 := f(b, b), \beta_1, \beta_2 \in \mathbb{N} F \text{ such that } \alpha_1^2 \beta_1^2 + \alpha_2^2 \beta_2^2 = 0,
\]
\[
\gamma = \frac{1}{2} (\alpha_1^* \beta_1 \beta_2 - \alpha_2^* \beta_1 \beta_2) = \alpha_1^* \beta_1 \beta_2 - \alpha_2^* \beta_1 \beta_2,
\]
\[
X := \frac{1}{2} ((\beta_1 \otimes 1_k)V_0 + (\beta_2 \otimes 1_k)V_0), \quad P_+ := X^* X, \quad P_- := XX^*.
\]
We consider $S(g)$ as an $E\star$-C*-subalgebra of $S(f)$ (Corollary 2.1.17 e)).

a) $ab = ba$, $\gamma^2 = -\alpha_1^* \alpha_2^*$. We put $c := ab = \omega(1,1)$.

b) $X, V_\omega, P_\omega \in S(g)_\omega$.

c) We have
\[
P_\pm = \frac{1}{2} (V_\omega \pm (\gamma \otimes 1_k)V_\omega) \in \text{PrS}(f), \quad P_+ + P_- = V_\omega, \quad P_+ P_- = 0,
\]
\[
X^2 = 0, \quad XP_+ = X, \quad P_- X = X, \quad P_+ X = XP_- = 0, \quad X + X^* \in \mathbb{N} S(f).
\]
d) The map 
\[ E \to P_+ S(f) P_+ \quad x \mapsto P_+ (x \otimes 1_k) P_+ \]

is an injective unital C*-homomorphism. We identify E with its image with respect to this map and consider \( P_+ S(f) P_+ \) as an E-C*-algebra.

e) The map 
\[ \varphi_\pm : S(g) \to P_\pm S(f) P_\pm, \quad Y \mapsto P_\pm Y P_\pm = P_\pm Y = Y P_\pm \]

is an injective unital C*-homomorphism. If \( Y_1, Y_2 \in \text{Un}(g) \) then \( \varphi_\pm Y_1 + \varphi_\pm Y_2 \in \text{Un} S(f) \).

f) The map 
\[ \psi : S(f) \to S(f), \quad Z \mapsto (X + X^*)Z(X + X^*) \]

is an E-C**-isomorphism such that 
\[ \psi^{-1} = \psi, \psi(P_+ S(f) P_+) = P_- S(f) P_-, \quad \psi \varphi_+ = \varphi_-, \quad \psi \varphi_- = \varphi_+. \]

If \( Y_1, Y_2 \in S(g) \) then 
\[ \varphi_\pm Y_1 + \varphi_\pm Y_2 = (\varphi_\pm Y_1 + \varphi_- V_1) \psi(\varphi_\pm Y_2 + \varphi_- V_1). \]

g) If \( p \in \text{Pr}(g) \) then 
\[ (X(\varphi_+ p))(X(\varphi_+ p))^* = \varphi_+ p, \quad (X(\varphi_+ p))(X(\varphi_+ p))^* = \varphi_- p. \]

h) Let \( R \) be the subgroup \( \{1, a, b, c\} \) of \( T, h := f(R \times R), d \in T \) such that \( d = N \) and \( d_i = a_i \) for every \( i \in N \), and 
\[ a = f(d, d), \quad a' = f_{2n-1}(2n-1, 2n-1), \quad a'' = f_{2n}(2n, 2n). \]

Then \( a_1 = aa', a_2 = aa'', -a'a'' = (a' a'')^2 \),

<table>
<thead>
<tr>
<th>h</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>aa'</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>-a</td>
<td>aa''</td>
<td>-a''</td>
</tr>
<tr>
<td>c</td>
<td>-a'</td>
<td>a''</td>
<td>-a'a''</td>
</tr>
</tbody>
</table>

is the table of \( h, P_\pm \in \text{Pr}(S(h)) \), and the map
\[ \varphi : S(h) \to E_{2,2}, \quad Z \mapsto \begin{bmatrix} Z_0 + \gamma^* Z_c & \alpha a Z_0 - \alpha a' Z_b \\ Z_0^* + \alpha a' Z_0 - \alpha a Z_b & Z_0 - \gamma Z_c \end{bmatrix} \]

is an E-C**-isomorphism. In particular
\[ \varphi P_+ = \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix}, \quad \varphi P_- = \begin{bmatrix} 0 & 0 \\ 0 & 1_k \end{bmatrix} \]

and \( E_{2,2} \) is E-C**-isomorphic to an E-C**-subalgebra of \( S(f) \).

i) Assume \( I = \mathbb{N}_{2n} \) and \( T_{n-1} = T_{2n} = Z_2 \). Then \( T \approx S 	imes Z_2 \times Z_2 \), \( \varphi \pm \) is an E-C*-isomorphism with inverse
\[ P \pm S(f) P \pm \to S(f_0), \quad Z \mapsto 2 \sum_{i \in T_0} (Z_i \otimes 1_k) V_i, \]

and \( S(f) \cong S(g)_{2,2} \)
a) is easy to see.
b) follows from Proposition 4.1.3 b).
c) follows from a) and Theorem 2.2.18 b), h).
d) follows from Theorem 2.2.18 c).
e) By b) and c), the map is well-defined. The assertion follows now from Theorem 2.2.18 d), h).
f) follows from b), c), and Theorem 2.2.18 h).
g) follows from b) and Proposition 2.2.17 d).
h) follows from c), d), Proposition 3.2.1 a), Corollary 3.2.2 d), and Proposition 3.2.3 c).
i) follows from Theorem 2.2.18 f).

PROPOSITION 4.1.8. We use the notation and the hypotheses of Theorem 4.1.7 and assume \( I := \mathbb{N}_2 \), \( T_1 := \mathbb{Z}_2 \), and \( T_2 := \mathbb{Z}_{2m} \) with \( m \in \mathbb{N} \).

a) \( a = (1, 0), b = (0, m), c = (1, m), \alpha = 1_E, \alpha' = f_1(1, 1), \alpha'' = f_2(m, m), \) and 

\[
P_\pm S(f)P_\pm = \{ (x \otimes 1_k)P_\pm \mid x \in E \}.
\]

b) If \( m = 1 \) then there are \( \alpha, \beta, \gamma, \delta \in \mathbb{U}n Ec \) such that \( f \) is given by the following table:

<table>
<thead>
<tr>
<th>( f )</th>
<th>( [0, 1] )</th>
<th>( [0, 2] )</th>
<th>( [0, 3] )</th>
<th>( [1, 0] )</th>
<th>( [1, 1] )</th>
<th>( [1, 2] )</th>
<th>( [1, 3] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [0, 1] )</td>
<td>( a )</td>
<td>( \beta )</td>
<td>( \gamma )</td>
<td>( -1_E )</td>
<td>( -\alpha )</td>
<td>( -\beta )</td>
<td>( -\gamma )</td>
</tr>
<tr>
<td>( [0, 2] )</td>
<td>( \beta )</td>
<td>( \alpha' \beta \gamma )</td>
<td>( \alpha' \gamma )</td>
<td>( -1_E )</td>
<td>( -\beta )</td>
<td>( -\alpha' \beta \gamma )</td>
<td>( -\alpha' \gamma )</td>
</tr>
<tr>
<td>( [0, 3] )</td>
<td>( \gamma )</td>
<td>( \alpha' \gamma )</td>
<td>( \beta' \gamma )</td>
<td>( -1_E )</td>
<td>( -\gamma )</td>
<td>( -\alpha' \gamma )</td>
<td>( -\beta' \gamma )</td>
</tr>
<tr>
<td>( [1, 0] )</td>
<td>( 1_E )</td>
<td>( 1_E )</td>
<td>( 1_E )</td>
<td>( \delta )</td>
<td>( \delta )</td>
<td>( \delta )</td>
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</tr>
<tr>
<td>( [1, 1] )</td>
<td>( a )</td>
<td>( \beta )</td>
<td>( \gamma )</td>
<td>( -\delta )</td>
<td>( -\alpha \delta )</td>
<td>( -\beta \delta )</td>
<td>( -\gamma \delta )</td>
</tr>
<tr>
<td>( [1, 2] )</td>
<td>( \beta )</td>
<td>( \alpha' \beta \gamma )</td>
<td>( \alpha' \gamma )</td>
<td>( -\delta )</td>
<td>( -\beta \delta )</td>
<td>( -\alpha' \beta \gamma \delta )</td>
<td>( -\alpha' \gamma \delta )</td>
</tr>
<tr>
<td>( [1, 3] )</td>
<td>( \gamma )</td>
<td>( \alpha' \gamma )</td>
<td>( \beta' \gamma )</td>
<td>( -\delta )</td>
<td>( -\gamma \delta )</td>
<td>( -\alpha' \gamma \delta )</td>
<td>( -\beta' \gamma \delta )</td>
</tr>
</tbody>
</table>

c) We assume \( \mathbb{k} := \mathbb{C} \) and \( m := 1 \) and put for all \( j, k \in \{ 0, 1 \} \)

\[
\phi_{jk} : S(f) \rightarrow E, \quad Z \mapsto (Z_0 + (-1)^j Z_b + i^j Z_{(k, 1)} - i^j Z_{(k, 3)}),
\]

\[
\phi : S(f) \rightarrow E^4, \quad Z \mapsto (\phi_{00} Z, \phi_{01} Z, \phi_{10} Z, \phi_{11} Z).
\]

If we take \( \alpha := \beta := \gamma := -\delta := \beta_1 = \beta_2 := 1_E \) in b) then the map

\[
S(f) \rightarrow E_{2,2} \times E^4, \quad Z \mapsto \left( \begin{array}{cc} Z_0 + Z_{(1,2)} & Z_{(1,0)} - Z_0 \\ Z_{(1,0)} + Z_b & Z_0 - Z_{(1,2)} \end{array} \right), \quad \phi Z
\]

is an E-C**-isomorphism.

a) Use Corollary 4.1.2 and Proposition 2.1.2 b).
b) Use Proposition 6437 a) and Proposition 4.1.1.
c) follows from b) and Proposition 3.4.1 f).

4.2 A special case

Throughout this section we denote by \( S \) a totally ordered set, put \( T := (\mathbb{Z}_2)^S \), and fix a map \( \rho : S \rightarrow Un E' \). We define for every \( s \in S \), \( f_s \in F(\mathbb{Z}_2, E) \) by putting \( f_s(1, 1) = \rho(s) \) (Proposition 3.1.1 a)). Moreover we denote by \( f_\rho \) the Schur function \( f \) defined in Proposition 4.1.1 b) (with \( I \) replaced by \( S \)) and put \( Cl(\rho) := S(f_\rho) \).
Lemma 4.2.1. \( \Psi_f(S) \) endowed with the composition law

\[ \Psi_f(S) \times \Psi_f(S) \to \Psi_f(S), \quad (A, B) \mapsto A \triangle B = (A \setminus B) \cup (B \setminus A) \]

is a locally finite commutative group (Definition 2.1.18) with \( \varepsilon \) as neutral element and the map

\[ \Psi_f(S) \to T, \quad A \mapsto \varepsilon_A \]

is a group isomorphism with inverse

\[ T \to \Psi_f(S), \quad x \mapsto \{ s \in S \mid x(s) = 1 \} \].

We identify \( T \) with \( \Psi_f(S) \) by using this isomorphism and write \( s \) instead of \( [s] \) for every \( s \in S \). For \( A, B \in T \),

\[ f_\varepsilon(A, B) = (-1)^{\tau} \prod_{s \in A \setminus B} \rho(s), \]

where \( \tau \) is defined in Proposition 4.1.1 b).

Proposition 4.2.2. Assume \( S \) finite and let \( F \) be an E-C*-algebra. Let further \((x_s)_{s \in S}\) be a family in \( F \) such that for all distinct \( s, t \in S \) and for every \( y \in E \),

\[ x_s x_t = -x_t x_s, \quad x_s^2 = \rho(s) \xi_F, \quad x_s^* = \rho(s)^* x_s, \quad x_s y = y x_s. \]

Then there is a unique E-C*-homomorphism \( \varphi : \text{Cl}(\rho) \to F \) such that \( \varphi V_s = x_s \) for all \( s \in S \). If the family \((\prod_{s \in A} x_s)_{A \subseteq S}\) is E-linearly independent (resp. generates \( F \) as an E-C*-algebra) then \( \varphi \) is injective (resp. surjective).

Put

\[ \varphi V_A := x_{s_1} x_{s_2} \cdots x_{s_n} \]

for every \( A = \{ s_1, s_2, \ldots, s_m \} \), where \( s_1 < s_2 < \cdots < s_m \) and

\[ \varphi : \text{Cl}(\rho) \to F, \quad X \mapsto \sum_{A \subseteq S} X_A \varphi V_A. \]

It is easy to see that \((\varphi V_s)(\varphi V_t) = \varphi (V_s V_t)\) and \( y \varphi V_s = (\varphi V_s) y\) for all \( s, t \in S \) and \( y \in E \) (Proposition 2.1.2 b)). Let

\[ A = \{ s_1, s_2, \ldots, s_m \} \subseteq S, \quad B = \{ t_1, t_2, \ldots, t_n \} \subseteq S, \quad \{ r_1, r_2, \ldots, r_p \} = A \triangle B, \]

where the letters are written in strictly increasing order. Then

\[ (\varphi V_A)(\varphi V_B) = x_{s_1} x_{s_2} \cdots x_{s_n} x_{t_1} x_{t_2} \cdots x_{t_n} = f_\varepsilon(A, B) x_{r_1} x_{r_2} \cdots x_{r_p} = \]

\[ = f_\varepsilon(A, B) \varphi V_{A \triangle B} = \varphi (f_\varepsilon(A, B) \Box 1_K)V_{A \triangle B} = \varphi (V_A V_B), \]

\[ (\varphi V_A)^* = x_{s_n}^* \cdots x_{s_2}^* x_{s_1}^* = (-1)^{\binom{m+1}{2}} x_{s_n}^* x_{s_2}^* \cdots x_{s_1}^* = \]

\[ = (-1)^{\binom{n+1}{2}} \prod_{i \in N_{s_n}} \rho(s_i)^* x_{s_i} x_{s_2} \cdots x_{s_n} = (-1)^{\binom{n+1}{2}} \prod_{i \in N_{s_n}} \rho(s_i)^* \varphi V_A = \]

\[ = \varphi ((-1)^{\binom{n+1}{2}} \prod_{i \in N_{s_n}} \rho(s_i)^* \Box 1_K)V_A = \varphi (V_A^*). \]

by Proposition 4.1.3 e).
For $X, Y \in \text{Cl}(\rho)$ (by Theorem 2.1.9 c), g),
\[
(\varphi X)(\varphi Y) = \left( \sum_{A \in T} X_A \varphi V_A \right) \left( \sum_{B \in T} Y_B \varphi V_B \right) = \sum_{A, B \in T} X_A Y_B (\varphi V_A)(\varphi V_B) = \\
= \sum_{A, B \in T} X_A Y_B \varphi(V_A V_B) = \sum_{A, B \in T} X_A Y_B \varphi f(A, B) \varphi V_A V_B = \\
= \sum_{A \in T} \sum_{C \in T} X_A Y_A \varphi f(A, A \Delta C) \varphi V_C = \sum_{C \in T} \left( \sum_{A \in T} f(A, A \Delta C) X_A \varphi V_C \right) \varphi V_C = \\
= \sum_{C \in T} (XY)_C \varphi V_C = \varphi(XY),
\]

(Proposition 4.1.3 e) i.e. $\varphi$ is an $E$-C*-homomorphism. The uniqueness and the last assertions are obvious (by Theorem 2.1.9 a)).

**Proposition 4.2.3** Let $m, n \in \mathbb{N} \cup \{0\}$, $S = \mathbb{N}_{2n}$, $S' = \mathbb{N}_{2n+1}$, $K' = \mathbb{P}(\mathbb{S}')$, $(\alpha_i)_{i \in \mathbb{N}} \in (\text{UnE}^\infty)^m$,
\[
\rho' : S' \rightarrow \text{Un}_n E^\infty, \quad s \mapsto \left\{ \begin{array}{ll}
\rho(s) & \text{if } s \in S \\
\alpha_i^2 f_\rho(s) & \text{if } s = 2n + i \text{ with } i \in \mathbb{N}_m
\end{array} \right.
\]
and $A_i := A \cup \{2n + i\}$ for every $A \subseteq T$ and $i \in \mathbb{N}_m$.

a) $i \in \mathbb{N}_m \Rightarrow \tilde{f}_\rho(S_i) = \alpha_i^2$,
\[
(V_S^i) = (\alpha_i^2 \otimes 1_K) V_i^P.
\]
b) $P = \frac{1}{4} V_i^P + \frac{1}{2} \sum_{i \in \mathbb{N}_m} (\alpha_i^2 \otimes 1_K) V_i^P \in \text{PrCl}(\rho')$.
c) There is a unique injective $E$-C*-homomorphism $\varphi : \text{Cl}(\rho) \rightarrow \text{P CI}(\rho') P$ such that
\[
\varphi V_i^P = PV_i^P P = PV_i^P = V_i^P P
\]
for every $s \in S$.
d) If $m \in \mathbb{N}_2$ then $\varphi$ is an $E$-C*-isomorphism.

a) By Proposition 4.1.3 e),
\[
\tilde{f}_\rho(S_i) = (-1)^{n(2n+1)} \prod_{\alpha \in \mathbb{S}_n} \rho(s)^* = \left( (-1)^{n(2n+1)} \prod_{\alpha \in \mathbb{S}_n} \rho(s)^* \right) \alpha_i^2 f_\rho(S) = \alpha_i^2,
\]
\[
(V_S^i) = (\alpha_i^2 \otimes 1_K) V_i^P.
\]
b) follows from a) and Corollary 4.1.6.
c) By Proposition 4.1.3 c), for $s \in S$, $V_S^i V_S^\perp = V_S^\perp V_S^i$ for every $i \in \mathbb{N}_m$ so $V_i^P P = PV_i^P$. By b), for distinct $s, t \in S$ (Proposition 4.1.3 b),
\[
(\varphi V_i^P)(\varphi V_i^P) = PV_i^P V_i^P = -PV_i^P V_i^P = -(\varphi V_i^P)(\varphi V_i^P),
\]
\[
(\varphi V_i^P)^2 = P(V_i^P)^2 = P(\rho(s) \otimes 1_K) V_i^P = (\rho(s) \otimes 1_K) P,
\]
\[
(\varphi V_i^P)^* = P(V_i^P)^* = P(\rho(s)^* \otimes 1_K) V_i^P = (\rho(s) \otimes 1_K)^* P.
\]
By Proposition 4.2.2 there is a unique $E$-C*-homomorphism $\varphi : \text{Cl}(\rho) \rightarrow \text{P CI}(\rho') P$ with the given properties.
Let $X \in \text{Cl}(\rho)$ with $\varphi X = 0$. Then

$$0 = \left( \sum_{A \in S} (X_A \otimes 1_K') V^\rho_A \right) p = \frac{1}{2} \sum_{A \in S} (X_A \otimes 1_K') V^\rho_A + \frac{1}{2m} \sum_{A \in S} \sum_{i \in I_A} (X_A \otimes 1_K) f'_{iA} (A, S_i) V^\rho_{A_S}$$

and this implies $X_A = 0$ for all $A \in S$ (Theorem 2.1.9 a)). Thus $\varphi$ is injective.

d) The case $m = 1$

Let $Y \in \text{PCI}(\rho)$. Then (by Proposition 2.1.2 b))

$$Y = YP = \frac{1}{2} Y + \frac{1}{2} \sum_{A \in S} (\alpha^1_A \otimes 1_K') Y_{S_A}$$

$$= \sum_{A \in S} ((\alpha^1_A f_p (S_1, A) Y_A) \otimes 1_K') V^\rho_{S_A}$$

$$+ \sum_{A \in S} ((\alpha^1_A f_p (S_1, A) Y_A) \otimes 1_K') V^\rho_{S_1 A}$$

so

$$\begin{align*}
Y_A &= \alpha^1_A f_p (S_1, (S \triangle A)) Y_{S \triangle A}, \\
Y_{A_1} &= \alpha^1_A f_p (S_1, S \triangle A) Y_{S \triangle A}
\end{align*}$$

for every $A \in S$. If we put

$$X = 2 \sum_{A \in S} (Y_A \otimes 1_K') V^\rho_A \in \text{Cl}(\rho)$$

then

$$\varphi X = \frac{1}{2} \varphi X + \sum_{A \in S} ((\alpha^1_A f_p (S_1, A) Y_A) \otimes 1_K') V^\rho_{S_A} =$$

$$= \sum_{A \in S} (Y_A \otimes 1_K') V^\rho_A + \sum_{A \in S} ((\alpha^1_A f_p (S_1, (S \triangle A)) Y_{S \triangle A}) \otimes 1_K') V^\rho_{A_1} =$$

$$= \sum_{A \in S} (Y_A \otimes 1_K') V^\rho_A + \sum_{A \in S} (Y_{A_1} \otimes 1_K') V^\rho_{A_1} = Y.$$

Thus $\varphi$ is surjective.

The case $m = 2$

Let $Y \in \text{PCI}(\rho)$. Then

$$\begin{align*}
Y &= PY = \frac{1}{2} Y + \frac{1}{2 \sqrt{2}} ((\alpha^1_A \otimes 1_K') V^\rho_{S_A} + (\alpha^2_A \otimes 1_K') V^\rho_{S_2}) Y \\
Y &= YP = \frac{1}{2} Y + \frac{1}{2 \sqrt{2}} Y ((\alpha^1_A \otimes 1_K') V^\rho_{S_A} + (\alpha^2_A \otimes 1_K') V^\rho_{S_2}),
\end{align*}$$

$$\sqrt{2} Y = (\alpha^1_A \otimes 1_K') V^\rho_{S_A} Y + (\alpha^2_A \otimes 1_K') V^\rho_{S_2} Y = (\alpha^1_A \otimes 1_K') V^\rho_{S_A} + (\alpha^2_A \otimes 1_K') V^\rho_{S_2}.$$

For every $B \subset S$ put

$$B_a := B \cup \{2n + 1\}, \quad B_b := B \cup \{2n + 2\}, \quad B_c := B \cup \{2n + 1, 2n + 2\}.$$
Then

\[ V_{S_1}^p Y = \sum_{BCS} ((Y_{bf}(S_1, B)) \otimes 1_{K'}) V_{S_1}^{p'} + \sum_{BCS} ((Y_{bf}(S_1, B_a)) \otimes 1_{K'}) V_{S_1}^{p'} + \]

\[ + \sum_{BCS} ((Y_{bf}(S_1, B_b)) \otimes 1_{K'}) V_{S_1}^{p'} + \sum_{BCS} ((Y_{bf}(S_1, B_c)) \otimes 1_{K'}) V_{S_1}^{p'} \]

\[ V_{S_2}^p Y = \]

\[ = \sum_{BCS} ((Y_{bf}(S_2, B)) \otimes 1_{K'}) V_{S_2}^{p'} + \sum_{BCS} ((Y_{bf}(S_2, B_a)) \otimes 1_{K'}) V_{S_2}^{p'} + \]

\[ + \sum_{BCS} ((Y_{bf}(S_2, B_b)) \otimes 1_{K'}) V_{S_2}^{p'} + \sum_{BCS} ((Y_{bf}(S_2, B_c)) \otimes 1_{K'}) V_{S_2}^{p'} \]

\[ YV_{S_1}^p = \sum_{BCS} ((Y_{bf}(B, S_1)) \otimes 1_{K'}) V_{S_1}^{p'} + \sum_{BCS} ((Y_{bf}(B_a, S_1)) \otimes 1_{K'}) V_{S_1}^{p'} + \]

\[ + \sum_{BCS} ((Y_{bf}(B_b, S_1)) \otimes 1_{K'}) V_{S_1}^{p'} + \sum_{BCS} ((Y_{bf}(B_c, S_1)) \otimes 1_{K'}) V_{S_1}^{p'} \]

\[ YV_{S_2}^p = \]

\[ = \sum_{BCS} ((Y_{bf}(B, S_2)) \otimes 1_{K'}) V_{S_2}^{p'} + \sum_{BCS} ((Y_{bf}(B_a, S_2)) \otimes 1_{K'}) V_{S_2}^{p'} + \]

\[ + \sum_{BCS} ((Y_{bf}(B_b, S_2)) \otimes 1_{K'}) V_{S_2}^{p'} + \sum_{BCS} ((Y_{bf}(B_c, S_2)) \otimes 1_{K'}) V_{S_2}^{p'} \]

\[ \sqrt{2} Y = \sum_{BCS} ((\alpha_1 Y_{bf}(S_1, B_a) + \alpha_2 Y_{bf}(S_2, B_b)) \otimes 1_{K'}) V_{S_2}^{p'} + \]

\[ + \sum_{BCS} ((\alpha_1 Y_{bf}(S_1, B) + \alpha_2 Y_{bf}(S_2, B_c)) \otimes 1_{K'}) V_{S_2}^{p'} + \]

\[ + \sum_{BCS} ((\alpha_1 Y_{bf}(S_1, B_c) + \alpha_2 Y_{bf}(S_2, B)) \otimes 1_{K'}) V_{S_2}^{p'} + \]

\[ + \sum_{BCS} ((\alpha_1 Y_{bf}(S_1, B) + \alpha_2 Y_{bf}(S_2, B_a)) \otimes 1_{K'}) V_{S_2}^{p'} \]

\[ \sqrt{2} Y = \sum_{BCS} ((\alpha_1 Y_{bf}(B, S_1) + \alpha_2 Y_{bf}(B_a, S_2)) \otimes 1_{K'}) V_{S_2}^{p'} + \]

\[ + \sum_{BCS} ((\alpha_1 Y_{bf}(B, S_1) + \alpha_2 Y_{bf}(B_c, S_2)) \otimes 1_{K'}) V_{S_2}^{p'} + \]

\[ + \sum_{BCS} ((\alpha_1 Y_{bf}(B, S_1) + \alpha_2 Y_{bf}(B, S_a)) \otimes 1_{K'}) V_{S_2}^{p'} + \]

\[ + \sum_{BCS} ((\alpha_1 Y_{bf}(B, S_1) + \alpha_2 Y_{bf}(B, S_b)) \otimes 1_{K'}) V_{S_2}^{p'} \]
It follows for $B \subset S$,

$$
\sqrt{2} Y_B = a_1 Y_{S \triangle B} (S \triangle B, S_1) + a_2 Y_{(S \triangle B),f} ((S \triangle B)_e, S_2),
$$

$$
\sqrt{2} Y_B = a_1 Y_{(S \triangle B),f} ((S \triangle B)_e, S_1) + a_2 Y_{(S \triangle B),f} ((S \triangle B)_g, S_2),
$$

$$
\sqrt{2} Y_B = a_1 Y_{(S \triangle B),f} ((S \triangle B)_g, S_1) + a_2 Y_{(S \triangle B),f} ((S \triangle B)_g, S_2),
$$

so by Proposition 4.1.3 a),b), $Y_B = 0$. If we put

$$
X := \sum_{B \subset S} (Y_B \otimes 1_k) V'_B \in Cl(\rho)
$$

then

$$
\varphi X = \left( 2 \sum_{B \subset S} (Y_B \otimes 1_k) V'_B \right) P =
$$

$$
= \sum_{B \subset S} (Y_B \otimes 1_k) V'_B + \frac{1}{\sqrt{2}} \sum_{B \subset S} \left( ((a_1 Y_{(B,S)}(B,S_1)) \otimes 1_k) V'_B \right)
$$

and so for $B \subset S$,

$$
(\varphi X)_B = Y_B, \quad (\varphi X)_B = \frac{1}{\sqrt{2}} a_1 Y_{S \triangle B} (S \triangle B, S_1) = Y_B,
$$

$$
(\varphi X)_B = \frac{1}{\sqrt{2}} a_2 Y_{S \triangle B} (S \triangle B, S_2) = Y_B, \quad (\varphi X)_B = 0 = Y_B.
$$

Thus $\varphi X = Y$ and $\varphi$ is surjective.

**Remark.** If $m = 3$ then $\varphi$ may be not surjective.

**PROPOSITION 4.2.4** Let $K := \mathbb{R}, n \in \mathbb{N} \cup \{0\}$, $S := \mathbb{N}_{2n}$, and

$$
\rho^\prime : \mathbb{N}_{2n+1} \rightarrow Un E^c, \quad s \mapsto \left\{ \begin{array}{ll}
\rho(s) & \text{if } s \in S \\
\rho^\prime(s) & \text{if } s = 2n + 1.
\end{array} \right.
$$

Let $\text{Cl}(\rho)$ be the complexification of $Cl(\rho)$, considered as a real $E$-$C^*$-algebra ([C1] Theorem 4.1.1.8 a)) by using the embedding

$$
E \rightarrow \text{Cl}(\rho), \quad x \mapsto (x \otimes 1_k) V_0 \rho, 0).
$$

Then there is a unique $E$-$C^*$-isomorphism $\varphi : Cl(\rho^\prime) \rightarrow \text{Cl}(\rho)$ such that $\varphi V'_s = (V'_s, 0)$ for every $s \in S$ and

$$
\varphi V'_s = (0, -(\bar{f}^\prime_{\rho}(S) \otimes 1_k) V'_s).
$$

We put

$$
X_s := \left\{ \begin{array}{ll}
(V'_s, 0) & \text{if } s \in S \\
(0, -(\bar{f}^\prime_{\rho}(S) \otimes 1_k) V'_s) & \text{if } s = 2n + 1.
\end{array} \right.
$$

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For \( s \in S \), by Proposition 4.1.3 b),
\[
x_{0}x_{2n+1} = (V_{g}^{p}, 0)(0, -\tilde{f}_{\rho}(S)\otimes 1_{K}) V_{g}^{p}
\]
\[
= (0, (\tilde{f}_{\rho}(S)\otimes 1_{K}) V_{g}^{p} V_{g}^{p}) = (0, (\tilde{f}_{\rho}(S)\otimes 1_{K}) V_{g}^{p}) (V_{g}^{p}, 0) = -x_{2n+1}x_{s}.
\]
By Proposition 2.1.2 b), e),
\[
x_{2n+1}^{2} = (-((\tilde{f}_{\rho}(S)\otimes 1_{K}) V_{g}^{p})^{2}, 0) =
\]
\[
= (\rho'(2n+1)\otimes 1_{K})(V_{g}^{p}, 0), \quad x_{2n+1} = (0, ((\tilde{f}_{\rho}(S)\otimes 1_{K}) V_{g}^{p})^{*}) =
\]
\[
= (0, (\tilde{f}_{\rho}(S)^{*}\otimes 1_{K}) (\tilde{f}_{\rho}(S)\otimes 1_{K}) V_{g}^{p}) = (\rho'(2n+1)^{*}\otimes 1_{K})x_{2n+1},
\]
and the assertion follows from Proposition 4.2.2.

**PROPOSITION 4.2.5** Let \( n \in \mathbb{N} \setminus \{0\} \), \( S := N_{n}, S' := N_{n+2}, K' := \rho'(\mathfrak{P}(S')) \), \( \alpha_{1}, \alpha_{2} \in UnE' \), and

\[
\rho' : S' \rightarrow Un E', \quad s \mapsto \begin{cases} 
\rho(s) & \text{if } s \in S \\
\alpha_{1}^{2} & \text{if } s = n+1 \\
-\alpha_{2}^{2} & \text{if } s = n+2
\end{cases}
\]

a) There is a unique \( E'-\text{C}^{*}\)-isomorphism \( \varphi : Cl(\rho') \rightarrow Cl(\rho)_{2,2} \) such that

\[
\varphi V_{g}^{p} = \begin{bmatrix} V_{g}^{p} & 0 \\
0 & -V_{g}^{p}
\end{bmatrix}
\]

for every \( s \in S \) and

\[
\varphi V_{n+1}^{p} = (\alpha_{1}\otimes 1_{K}) \begin{bmatrix} V_{g}^{p} & 0 \\
0 & V_{g}^{p}
\end{bmatrix}, \quad \varphi V_{n+2}^{p} = (\alpha_{2}\otimes 1_{K}) \begin{bmatrix} 0 & -V_{g}^{p} \\
V_{g}^{p} & 0
\end{bmatrix}.
\]

b)

\[
\varphi \frac{1}{2}(V_{g}^{p} - (\alpha_{1}^{2}\otimes 1_{K}) V_{g}^{p})_{n+1,n+2} = \begin{bmatrix} V_{g}^{p} & 0 \\
0 & 0
\end{bmatrix}.
\]

\[
\varphi \frac{1}{2}(V_{g}^{p} - (\alpha_{2}^{2}\otimes 1_{K}) V_{g}^{p})_{n+1,n+2} = \begin{bmatrix} 0 & 0 \\
0 & V_{g}^{p}
\end{bmatrix}.
\]

a) Put

\[
x_{i} := \begin{bmatrix} V_{g}^{p} & 0 \\
0 & -V_{g}^{p}
\end{bmatrix}
\]

for every \( s \in S \) and

\[
x_{n+1} := (\alpha_{1}\otimes 1_{K}) \begin{bmatrix} V_{g}^{p} & 0 \\
0 & V_{g}^{p}
\end{bmatrix}, \quad x_{n+2} := (\alpha_{2}\otimes 1_{K}) \begin{bmatrix} 0 & -V_{g}^{p} \\
V_{g}^{p} & 0
\end{bmatrix}.
\]

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For distinct \( s, t \in S \) and \( i \in \mathbb{N}_2 \),

\[
\begin{align*}
x_{s^2} &= -x_{t^2}, \\
x_s^2 &= (\rho'(s) \otimes 1_K)[V_A^0 \ 0 \\
&\quad \quad 0 \\
&\quad \quad V_A^0], \\
x_s'^2 &= (\rho'(s) \otimes 1_K)^*x_s.
\end{align*}
\]

By Proposition 4.2.2 there is a unique \( E-C^*\)-homomorphism \( \varphi : \text{Cl}(\rho') \to \text{Cl}(\rho)_{2S} \) satisfying the given conditions. We put for every \( A \subset S \) and \( i \in \mathbb{N}_2 \)

\[
|A| = \text{Card } A, \quad A_i = A \cup \{n + i\}, \quad A_3 = A \cup \{n, n + 1, n + 2\}.
\]

For \( A \subset S \),

\[
\varphi V_{A_i}^0 = (\alpha_1 \otimes 1_K)[V_A^0 \ 0 \\
&\quad \quad 0 \\
&\quad \quad V_A^0] = \\
&= (\alpha_1 \otimes 1_K)[0 \\
&\quad \quad (-1)^{|A|}V_A^0 \\
&\quad \quad 0], \\
\varphi V_{A_2}^0 = (\alpha_2 \otimes 1_K)[V_A^0 \ 0 \\
&\quad \quad 0 \\
&\quad \quad -V_A^0] = \\
&= (\alpha_2 \otimes 1_K)[0 \\
&\quad \quad (-1)^{|A|}V_A^0 \\
&\quad \quad 0], \\
\varphi V_{A_3}^0 = ((\alpha_1 \alpha_2) \otimes 1_K)[0 \\
&\quad \quad 0 \\
&\quad \quad -V_A^0] = \\
&= ((\alpha_1 \alpha_2) \otimes 1_K)[0 \\
&\quad \quad (-1)^{|A|}V_A^0].
\]

Then for \( Y \in \text{Cl}(\rho') \),

\[
\begin{align*}
(\varphi Y)_{11} &= \sum_{A \in S} (Y_A + (\alpha_1 \alpha_2)Y_{A_3}) \otimes 1_K)V_A^0, \\
(\varphi Y)_{12} &= \sum_{A \in S} (\alpha_2 Y_A - \alpha_1 Y_{A_2}) \otimes 1_K)V_A^0, \\
(\varphi Y)_{21} &= \sum_{A \in S} (-1)^{|A|}((\alpha_1 Y_A + \alpha_2 Y_{A_2}) \otimes 1_K)V_A^0, \\
(\varphi Y)_{22} &= \sum_{A \in S} (-1)^{|A|}(Y_A + \alpha_1 \alpha_2 Y_{A_3}) \otimes 1_K)V_A^0.
\end{align*}
\]

It follows from the above identities that \( \varphi \) is bijective.

b) By the above,

\[
\varphi V_{[n+1,n+2]}^0 = \varphi V_{A_3}^0 = ((\alpha_1 \alpha_2) \otimes 1_K)[V_A^0 \\
&\quad \quad 0 \\
&\quad \quad -V_A^0],
\]

and the assertion follows.
COROLLARY 4.2.6 Let $m, n \in \mathbb{N} \cup \{0\}$, $S := \mathbb{N}_n$, $(\alpha_i) \in \mathbb{N}_m \in (\text{Un } E)^{2m}$, and

$$
\rho' : \mathbb{N}_{n+2m} \to \text{Un } E^c, \quad s \mapsto \begin{cases} 
\rho(s) & \text{if } s \in S \\
-(1)^{\alpha_i^2} & \text{if } s = n + i.
\end{cases}
$$

Then $\text{Cl}(\rho') \cong \text{Cl}(\rho)_{2n \times 2n}$.

PROPOSITION 4.2.7 Let $\mathbb{K} := \mathbb{K}$, $n \in \mathbb{N} \cup \{0\}$, $S := \mathbb{N}_n$, $S' := \mathbb{N}_{n+2}$, $\alpha_1, \alpha_2 \in \text{Un } E^c$, and

$$
\rho' : S' \to \text{Un } E^c, \quad s \mapsto \begin{cases} 
\rho(s) & \text{if } s \in S \\
-\alpha_i^2 & \text{if } s = 2n + 1 \\
α_i & \text{if } s = 2n + 2
\end{cases}
$$

Then there is a unique $E$-$C^*$-isomorphism $\varphi : \text{Cl}(\rho') \to \text{Cl}(\rho) \otimes \mathbb{K}$ such that

$$
\varphi V_2^0 = \begin{cases} 
V_2^0 \otimes 1_{\mathbb{K}} & \text{if } s \in S \\
((\alpha_1 f_\rho(S) \otimes 1_K)V_2^0) \otimes i & \text{if } s = 2n + 1 \\
((\alpha_2 f_\rho(S) \otimes 1_K)V_2^0) \otimes j & \text{if } s = 2n + 2
\end{cases}
$$

where $i, j, k$ are the canonical unitaries of $\mathbb{K}$.

Put

$$
x_t := \begin{cases} 
V_2^0 \otimes 1_{\mathbb{K}} & \text{if } s \in S \\
((\alpha_1 f_\rho(S) \otimes 1_K)V_2^0) \otimes i & \text{if } s = 2n + 1 \\
((\alpha_2 f_\rho(S) \otimes 1_K)V_2^0) \otimes j & \text{if } s = 2n + 2
\end{cases}
$$

For distinct $s, t \in S$ and $l \in \mathbb{N}_2$, by Proposition 4.1.3 b),

$$
x_t x_s = x_t x_s^2 = (\rho'(s) \otimes 1_K)(V_2^0 \otimes 1_{\mathbb{K}}), \quad x_t^* = (\rho'(s) \otimes 1_K)^* x_t,
$$

$$
x_{2n+2} x_{2n+1} = x_{2n+2} x_{2n+1} x_{2n+2} = ((\alpha_1 f_\rho(S) \otimes 1_K)V_2^0) \otimes k = x_{2n+2} x_{2n+1},
$$

$$
(x_{2n+2})^2 = ((\alpha_1 f_\rho(S)^2) \otimes 1_K)(f_\rho(S) \otimes 1_K)V_2^0 \otimes (-1_{\mathbb{K}}) =
\begin{cases} 
(\rho'(2n + 1) \otimes 1_K)(V_2^0 \otimes 1_{\mathbb{K}}), & \text{if } s = 2n + 1 \\
(\rho'(2n + 2) \otimes 1_K)(V_2^0 \otimes 1_{\mathbb{K}}), & \text{if } s = 2n + 2
\end{cases}
$$

By Proposition 4.2.2 there is a unique $E$-$C^*$-homomorphism $\varphi : \text{Cl}(\rho') \to \text{Cl}(\rho) \otimes \mathbb{K}$ satisfying the given conditions. For $X \in \text{Cl}(\rho')$,

$$
\varphi X = \left( \sum_{A \subseteq S} (X_A \otimes 1_K)V_2^0 \right) \otimes 1_{\mathbb{K}} +
\begin{aligned}
+ \left( \sum_{A \subseteq S} (X_{A \cup \{2n+1\}} \alpha f_\rho(A.S) \otimes 1_K)V_{S \cup A} \right) \otimes i +
+ \left( \sum_{A \subseteq S} (X_{A \cup \{2n+2\}} \alpha f_\rho(A.S) \otimes 1_K)V_{S \cup A} \right) \otimes j +
+ \left( \sum_{A \subseteq S} (X_{A \cup \{2n+2\} A} \alpha f_\rho(S) A \otimes 1_K)V_{S \cup A} \right) \otimes k
\end{aligned}
$$

and so $\varphi$ is bijective. \hfill \blacksquare
PROPOSITION 4.2.8 Let \( n \in \mathbb{N} \cup \{0\}, S := \mathbb{N}_{2n}, A' := A \cup (2n + 1) \) for every \( A \subset S \),

\[
\rho' : S \rightarrow \mathbb{H}^n, \quad s \mapsto \begin{cases} 
\rho(s) & \text{if } s \in S \\
\tilde{f}(s) & \text{if } s = 2n + 1
\end{cases}
\]

\[
P_\pm := \frac{1}{2} (V_{\theta'} \pm V_{\delta'}), \text{ and } \theta_\pm : (\mathcal{O}_{A \subset S} \tilde{E}) \rightarrow (\mathcal{O}_{A \subset S} \tilde{E}) \text{ defined by}
\]

\[
(\theta_+ \xi)_A := \frac{1}{\sqrt{2}} \xi_A, \quad (\theta_- \xi)_A := \pm \frac{1}{\sqrt{2}} f_P (S \Delta A, S) \xi_{S \Delta A}
\]

for every \( \xi \in (\mathcal{O}_{A \subset S} \tilde{E}) \) and \( A \subset S \).

a) 

\[
\tilde{f}_{\rho'} (S') = 1_E, \quad \langle V_{\delta'} \rangle^2 = V_{\theta'}, \quad P_\pm = \operatorname{PrCl} (\rho') \quad \text{\( \in \mathcal{O}_{A \subset S} \tilde{E} \)}
\]

\[
P_+ + P_- = V_{\theta'}, \quad V_{\delta'} \in \operatorname{Cl} (\rho') \quad \text{\( \in \mathcal{O}_{A \subset S} \tilde{E} \)}, \quad V_{\delta'} P_\pm = \pm P_\pm
\]

b) For \( A \subset S \),

\[
f_P (A, S)^* = f_P (S', A)^* = f_P (S', (S \Delta A)^*)
\]

c) \( \theta_+ \in \mathcal{L}_E (\mathcal{O}_{A \subset S} \tilde{E}, \mathcal{O}_{A \subset S} \tilde{E}) \) and for \( \eta \in (\mathcal{O}_{A \subset S} \tilde{E}) \) and \( A \subset S \),

\[
(\theta_+ \eta)_A := \frac{1}{\sqrt{2}} (\eta_A \pm f_P (A, S)^* \eta_{(S \Delta A)^*}) = \sqrt{2} (P_\pm \eta)_A.
\]

d) \( \theta_+ \theta_+ \) is the identity map of \( \mathcal{O}_{A \subset S} \tilde{E} \).

e) \( \theta_+ \theta_+ = P_+ \).

f) For every \( A \subset S \),

\[
\theta_\pm V_{\delta'} \theta_\pm = V_{\delta'} P_\pm = P_\pm V_{\delta'} P_\pm = P_\pm V_{\delta'} P_\pm.
\]

g) For every closed ideal \( F \) of \( E \) the map

\[
\varphi : \operatorname{Cl} (\rho', F) \rightarrow P_\pm \operatorname{Cl} (\rho', F) P_\pm, \quad X \mapsto \theta_+ X \theta_+
\]

is an E-C*-isomorphism with inverse

\[
P_\pm \operatorname{Cl} (\rho', F) P_\pm \rightarrow \operatorname{Cl} (\rho, F), \quad Y \mapsto \theta_+^* Y \theta_+-
\]

and the map

\[
\psi : \operatorname{Cl} (\rho', F) \rightarrow \operatorname{Cl} (\rho, F) \times \operatorname{Cl} (\rho, F), \quad Y \mapsto (\theta_+ P_+, Y P_+, \theta_+ P_+ Y P_+, \theta_+ Y \theta_+)
\]

is an E-C*-isomorphism.

a) By Proposition 4.1.3 d),e), \( V_{\delta'} \in \operatorname{Cl} (\rho') \) \( ^{\ast } \),

\[
\tilde{f}_{\rho'} (S') = (-1)^{(2n+1)} \prod_{s \in S} \rho(s)^* = (-1)^{(2n+1)} \left( \prod_{s \in S} \rho(s)^* \right) (2n + 1)^* = 1_E,
\]

\[
(V_{\delta'} \rangle^*) = \tilde{f}_{\rho'} (S') V_{\delta'} = V_{\delta'}, \quad (V_{\delta'} \rangle^*)^2 = \tilde{f}(S')^* V_{\theta'} = V_{\theta'}.
\]

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so
\[ P_\pm \in \text{PrCl}(\rho)^\prime, \quad V_\pm^p P_\pm = \pm P_\pm. \]

b) By a), Proposition 4.1.3 c), d), Proposition 4.1.1 b), and Proposition 1.1.2 b),
\[ f_\rho(A, S)^* = f_\rho(A, S)^* = f_\rho(A, S)^* = f_\rho(S', A)^* = f_\rho(S', (S \triangle A)') = f_\rho(S', (S \triangle A)'). \]

c) For \( \xi \in \mathbb{AC} E \),
\[ (\theta \xi|\eta) = \sum_{\mathbb{AC}} \eta_A \frac{1}{\sqrt{2}} \xi_A + \sum_{\mathbb{AC}} \eta_A^* \frac{1}{\sqrt{2}} f_\rho(S \triangle A, S) \xi_{S \triangle A} = \]
\[ = \sum_{\mathbb{AC}} \eta_A \frac{1}{\sqrt{2}} \xi_A + \sum_{\mathbb{AC}} \eta_A^{(S \triangle A)'} \frac{1}{\sqrt{2}} f_\rho(A, S) \xi_A = \]
\[ = \sum_{\mathbb{AC}} \frac{1}{\sqrt{2}} (\eta_A \pm f_\rho(A, S)^* \eta_{(S \triangle A)'}) \xi_A \]
so \( \theta \in \mathcal{L}_E(\mathbb{AC} E, \mathbb{AC} E) \)
and
\[ (\theta^* \eta) = \frac{1}{\sqrt{2}} (\eta_A \pm f_\rho(A, S)^* \eta_{(S \triangle A)'}). \]
By a) and b),
\[ (P_\pm \eta)_A = \frac{1}{2} \eta_A + \frac{1}{2} f_\rho(S', (S \triangle A)') \eta_{(S \triangle A)''} = \]
\[ = \frac{1}{2} (\eta_A \pm f_\rho(A, S)^* \eta_{(S \triangle A)'}) = \frac{1}{\sqrt{2}} (\theta^* \eta)_A. \]

d) For \( \xi \in \mathbb{AC} E \) and \( A \subset S \), by c),
\[ (\theta^* \eta \xi)_A = \frac{1}{\sqrt{2}} ((\theta \xi)_A \pm f_\rho(A, S)^*(\theta \xi))_{S \triangle A'} = \]
\[ = \frac{1}{2} (\xi_A + f_\rho(A, S)^* f_\rho(A, S) \xi_A) = \xi_A. \]

e) For \( \eta \in \mathbb{AC} E \) and \( A \subset S \), by b) and c),
\[ (\theta \eta \theta^* \eta)_A = \frac{1}{\sqrt{2}} (\theta^* \eta)_A = (P_\pm \eta)_A, \]
\[ (\theta \eta \theta^* \eta)_E = \pm \frac{1}{\sqrt{2}} f_\rho(S \triangle A, S) (\theta^* \eta)_{S \triangle A} = \]
\[ = \pm \frac{1}{2} f_\rho(S \triangle A, S) (\eta_{S \triangle A} \pm f_\rho(S \triangle A, S)^* \eta_A) = \pm \frac{1}{2} f_\rho(S \triangle A, S) \eta_{S \triangle A} + \frac{1}{2} \eta_A = \]
\[ = \frac{1}{2} (\eta_A \pm f_\rho(S', (S \triangle A)') \eta_{S \triangle A}) = \frac{1}{2} (V_\rho^p \eta)_{A'} \pm (V_\rho^p \eta)_{A'} = (P_\pm \eta)_{A'}, \]
so \( \theta \eta \theta^* = P_\pm \).

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f) For $\eta \in \bigotimes_{B \in S} \hat{E}$ and $B \subset S$, by a), b), c), e) and Proposition 4.1.1 b) (and Corollary 2.1.17 e)),

\[
(V^\eta_A P \pm \eta)_{B} = f_\eta (A, A \Delta B)(P \pm \eta)_{A \Delta B} = f_\eta (A, A \Delta B)(\theta_\pm \eta)_{A \Delta B} =
\]

\[
= \frac{1}{\sqrt{2}} f_\eta (A, A \Delta B)(\theta_\pm \eta)_{A \Delta B} = \frac{1}{\sqrt{2}} (V^\eta_A \theta_\pm \eta)_{B} = (\theta_\pm V^\eta_A \theta_\pm \eta)_{B},
\]

\[
(\theta_\pm V^\eta_A \theta_\pm \eta)_{B} = \pm \frac{1}{\sqrt{2}} f_\eta (S \Delta B, S)(V^\eta_A \theta_\pm \eta)_{S \Delta A \Delta B} =
\]

\[
= \pm f_\eta (S \Delta B, S)f_\eta (A, S \Delta AA \Delta B)(P \pm \eta)_{S \Delta A \Delta B} = \pm f_\eta (S \Delta B, S)(V^\eta_A P \pm \eta)_{S \Delta B} =
\]

\[
= \pm (V^\eta_A V^\eta_B P \pm \eta)_{B} = (V^\eta_A P \pm \eta)_{B}.
\]

so by a),

\[
\theta_\pm V^\eta_A \theta_\pm = V^\eta_A P \pm = P \pm V^\eta_A P \pm = P \pm V^\eta_A.
\]

\[f_\eta \text{ is a surjective } E-C^*\text{-homomorphism. For } Y \in \text{Ker}\psi,
\]

\[
\theta^*_+ Y \theta_+ = \theta^*_+ Y \theta_- = 0,
\]

so by a) and e),

\[
P_+ Y = P_- Y = 0
\]

and we get

\[
Y = P_+ Y + P_- Y = 0
\]

i.e. $\psi$ is injective.

\[\square\]

REFERENCES


COMPETING INTERESTS

The authors declare no competing interests.