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Authors: swayamprabha tiwari[1], Dr. Sameena Saba[2]
Affiliations: integral university, lucknow[1], integral university, lucknow[2]
Orcid ids: 0000-0003-0304-9996[1]
Contact e-mail: swayam6288@gmail.com
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A Study of Generalized Partial Mock Theta Functions

1Swayamprabha Tiwari and 2Dr. Sameena Saba

1,2Department of Mathematics and Statistics,
Faculty of Science,
Integral University, Lucknow-226026, India
E-mails: 1e-mail: swayam6288@gmail.com
2e-mail: sameena9554@gmail.com

Abstract:
Generalization of partial sixth and third order mock theta functions is given and shown that these
generalized partial mock theta functions are $F_q$ - functions. q-integral representation of these
generalized partial mock theta functions is also given.

Keywords and Phrases:
Basic hypergeometric series, partial mock theta functions, q-integral.

1 Introduction:
S.Ramanujan in his last letter to G.H. Hardy[11, pp 354-355] introduced seventeen functions
whom he called mock theta functions, as they were not theta functions. He Stated two conditions
for a function to be a mock theta function:
(0) For every root of unity $\zeta$, there is $\theta$- function $\theta_\zeta(q)$ such that difference $f(q) - \theta_\zeta(q)$ is bounded
as $q \to \zeta$ radially.
(1) There is no single $\theta$-function which works for all $\zeta$ i.e., for every $\theta$-function $\theta(q)$ there is some
root of unity $\zeta$ for which difference $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ radially.

Of the seventeen mock theta functions, four were of third order, ten of fifth order in two groups
with five functions in each group and three of seventh order. Ramanujan did not specify what
he meant by the order of a mock theta function. Later Watson [12] added three more third order
mock theta functions, making the four third order mock theta functions to seven.

G.E. Andrews [13] while visiting Trinity College Cambridge University discovered some notebooks
of Ramanujan, and called it the "Lost" Notebook. In the Notebook Andrews found more theta
functions and some identities and Andrews and Hickeson [18] called them sixth order.

The partial sixth order mock theta functions of Ramanujan are:

$$\phi_{0,N}(q) = \sum_{n=0}^{N} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(-q)_{2n}}$$

$$\psi_{0,N}(q) = \sum_{n=0}^{N} \frac{(-1)^n q^{(n+1)^2} (q; q^2)_n}{(-q)_{2n+1}}$$

$$\rho_{0,N}(q) = \sum_{n=0}^{N} q^{\frac{n(n+1)}{2}} (-q)_n (q; q^2)_n$$
\[ \sigma_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{\frac{n(n+1)}{2}} (-q)^n}{(-q;q^2)_n}, \]

\[ \lambda_{0,N}(q) = \sum_{n=0}^{N} \frac{(-1)^n q^n (q^2)_n}{(-q)_n}, \]

\[ \mu_{0,N}(q) = \sum_{n=0}^{N} \frac{(-1)^n (q^2)_n}{(-q)_n}, \]

\[ \gamma_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2} (q)_n}{(q^3;q^3)_n}. \]

The partial third order mock theta functions of Ramanujan are:

\[ f_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{(-q)_n^2}, \]

\[ \phi_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{(-q^2;q^2)_n}, \]

\[ \psi_{0,N}(q) = \sum_{n=1}^{N} \frac{q^{n^2}}{(q^2;q^2)_n}, \]

\[ \chi_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{(1 - q + q^2) \cdots (1 - q^n + q^{2n})}, \]

\[ \omega_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{2n(n+1)}}{(q;q^2)^2_{n+1}}, \]

\[ \upsilon_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}. \]
\[ \rho_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{2n(n+1)}}{(1 + q + q^2)\ldots(1 + q^{2n+1} + q^{4n+2})}. \]

We give a generalization of the partial sixth order and partial third order mock theta functions. The generalized partial sixth order mock theta functions are:

\[ \phi_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} (-1)^n t^n q^{n(n-3) + n\alpha} z^{2n} \left( \frac{z^2}{q}; q \right)_n \left( \frac{z^2}{q^2}; q \right)_n, \]

\[ \psi_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} (-1)^n t^n q^{n(n-1) + n\alpha} z^{2n+1} \left( \frac{z^2}{q}; q \right)_{2n+1}, \]

\[ \rho_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{t^n q^{n(n-1) + n\alpha} z^{2n}}{\left( \frac{z^2}{q}; q \right)_{n+1}}, \]

\[ \sigma_{0,N}(t, \alpha, z; q) = \frac{1}{2(t)_\infty} \sum_{n=0}^{N} \frac{t^n q^{n(n-1) + n\alpha} z^{2n+1}}{\left( \frac{z^2}{q}; q \right)_{n+1}}, \]

\[ \lambda_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{t^n q^{n(n-1) + n\alpha} z^{2n}}{\left( \frac{z^2}{q}; q \right)_n}, \]

\[ \mu_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{t^n q^{n(n-1) + n\alpha} z^{2n}}{\left( \frac{z^2}{q}; q \right)_n}, \]

and

\[ \gamma_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{t^n q^{n(n-3) + n\alpha} z^{2n}}{\left( \frac{v^2 z}{q}; q \right)_n \left( \frac{v^4 z}{q}; q \right)_n}, \]

For \( t = 0, \alpha = 1 \), we have the generalized partial functions of Choi[15]. The generalized partial third order mock theta function are:

\[ f_{0,N}(t, \alpha, \beta, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{t^n q^{n^2 - 4n + n\beta} z^{2n}}{\left( -z; q \right)_n \left( \frac{-\alpha z}{q}; q \right)_n}, \]

\[ \phi_{0,N}(t, \alpha, \beta, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{t^n q^{n^2 - 3n + n\beta} z^{2n}}{\left( \frac{-\alpha z}{q}; q^2 \right)_n}. \]
\[ \psi_{0,N}(t, \alpha, \beta, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n q^{n^2 - n + n\beta} z^{n+1}}{\left(\frac{q^2}{q}; q^2\right)_{n+1}} \]

\[ \nu_{0,N}(t, \alpha, \beta, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n q^{n^2 - 2n + n\beta} z^{2n}}{\left(\frac{q^2}{q^2}; q^2\right)_{n+1}} \]

\[ \omega_{0,N}(t, \alpha, \beta, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n q^{2n^2 - 5n - 4 + n\beta} z^{2(n+1)}}{\left(\frac{q^2}{q^2}; q^2\right)_{n+1}(\frac{q^2}{q^2}; q^2)_{n+1}} \]

\[ \chi_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n q^{n^2 - 3n + n\beta} z^{2n}}{(vz; q)_n (-v^2z; q)_n} \]

and

\[ \rho_{0,N}(t, \alpha, \beta, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(t)_n q^{2n^2 - 3n + n\beta} z^{4n}}{(\frac{q^2}{q^2}; q^2)_{n+1}(\frac{q^2}{q^2}; q^2)_{n+1}} \]

where

\[ v = e^{\frac{\pi i}{3}} \]

For \( \beta = 1 \) and \( z = q \) we have a generalized five third order partial mock theta functions namely \( f_{0,N}, \phi_{0,N}, \psi_{0,N}, \omega_{0,N}, \omega_{0,N} \) of Andrews[1]. For \( t = 0, \beta = 1, \alpha = q \) the generalized functions \( f_{0,N}, \phi_{0,N}, \psi_{0,N} \) and \( \chi_{0,N} \) reduce to the third order mock theta functions of Ramanujan and \( \omega_{0,N}, \nu_{0,N} \) and \( \rho_{0,N} \) to the third order mock theta functions of Watson[12].

In this study we will show that these generalized functions are \( F_q \)-functions.

2 Notation:

We shall use the following usual basic hypergeometric notations:

For \( |q^k| \leq 1 \) \( a; q^k \) = \((1-a)(1-aq)(1-aq^2)...(1-aq^{n-1})\), \( n \geq 1 \)

\( (a; q^k)_0 = 1, \ (a; q^k)_{\infty} = \prod_{j=0}^{\infty} (1-az^j) \)

\[ A^{\phi_{A-1}}[a_1, a_2, \ldots, a_A; b_1, b_2, \ldots, b_{A-1}; q, z] \]

\[ = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \ldots (a_A; q)_n z^n}{(b_1; q)_n \ldots (b_{A-1}; q)_n (q; q)_n}, \ |z| < 1 \]
3 The Generalized Functions are $F_q$ - Functions:

We show that these partial generalized functions are $F_q$ - Function.

**Theorem 1**

The generalized partial sixth order mock theta functions $\phi_{0,N}(t, \alpha, z; q), \psi_{0,N}(t, \alpha, z; q), \rho_{0,N}(t, \alpha, z; q), \sigma_{0,N}(t, \alpha, z; q)$ and the generalized partial third order mock theta functions $f_{0,N}(t, \alpha, \beta, z; q)$, $\phi_{0,N}(t, \alpha, \beta, z; q), \psi_{0,N}(t, \alpha, \beta, z; q), \chi_{0,N}(t, \alpha, z; q), \rho_{0,N}(t, \beta, z; q), \omega_{0,N}(t, \alpha, \beta, z; q)$ are $F_q$ - Functions.

Proof:

We shall give the proof for $\phi_{0,N}(t, \alpha, z; q)$ only. The proofs for the other generalized partial mock theta functions are similar, hence omitted.

Applying the difference operator $D_{q,t}$ to $\phi_{0,N}(t, \alpha, z; q)$, we have:

$$D_{q,t} \phi_{0,N}(t, \alpha, z; q) = \phi_{0,N}(t, \alpha, z; q) - \phi_{0,N}(tq, \alpha, z; q)$$

$$= \frac{1}{(tq)_{\infty}} \sum_{n=0}^{N} \frac{(-1)^{n}(t)_{n}q^{n(n-3)+n\alpha}z^{2n}(\frac{z^2}{q}; q^2)_{n}}{(\frac{z^2}{q}; q)_{2n}} - \frac{1}{(tq)_{\infty}} \sum_{n=0}^{N} \frac{(-1)^{n}(tq)_{n}q^{n(n-3)+n\alpha}z^{2n}(\frac{z^2}{q}; q^2)_{n}}{(\frac{z^2}{q}; q)_{2n}}$$

$$= \frac{1}{(t)_{\infty}} \sum_{n=0}^{N} \frac{(-1)^{n}(t)_{n}q^{n(n-3)+n\alpha}z^{2n}(\frac{z^2}{q}; q^2)_{n}}{(\frac{z^2}{q}; q)_{2n}} - \frac{1}{(t)_{\infty}} \sum_{n=0}^{N} \frac{(-1)^{n}(tq)_{n}q^{n(n-3)+n\alpha}z^{2n}(\frac{z^2}{q}; q^2)_{n}(1 - tq^n)}{(\frac{z^2}{q}; q)_{2n}}$$

$$= \frac{1}{(t)_{\infty}} \sum_{n=0}^{N} \frac{(-1)^{n}(t)_{n}q^{n(n-3)+n\alpha+1}z^{2n}(\frac{z^2}{q}; q^2)_{n}}{(\frac{z^2}{q}; q)_{2n}}$$

$$= t\phi_{0,N}(t, \alpha + 1, z; q).$$

so

$$D_{q,t} \phi_{0,N}(t, \alpha, z; q) = \phi_{0,N}(t, \alpha + 1, z; q).$$

Hence $\phi_{0,N}(t, \alpha, z; q)$ is a $F_q$ - function.

As stated earlier the proofs for other partial generalized functions are similar, hence omitted.

4 Relation between the Generalized Partial Sixth Order Mock Theta Functions and Generalized Partial Third Order Mock Theta Functions:

**Theorem 2**

(i) $\phi_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_{\infty}} \sum_{n=0}^{N} \frac{(-1)^{n}(t)_{n}q^{n(n-3)+n\alpha}z^{2n}(\frac{z^2}{q}; q^2)_{n}}{(\frac{z^2}{q}; q)_{2n}} + \frac{z}{q} \psi_{0,N}(t, \alpha, z; q).$

(ii) $\sigma_{0,N}(t, \alpha, z; q) = \frac{z}{(1 + \frac{z}{q})D_{q,t} \rho_{0,N}(t, \alpha, z; q).}$

(iii) $D_{q,t} \phi_{0,N}(t, \alpha^2, \beta, z; q) = (1 + \frac{\alpha^2 z^2}{q^2}) \psi_{0,N}(t, \alpha, \beta, z; q).$
(iv) $\psi_{0,N}(t, \frac{-\alpha^2}{q}, \beta, z; q) = z D_{q,t} \nu_{0,N}(t, \alpha, \beta, z; q)$.

Proof of (i)

$\phi_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n \left( t \right)_n q^{n(2n-1)} + \alpha z^n \left( \frac{q}{q^2} \right)^n}{(\frac{z^2}{q}; q)_{2n+1}}$

$= \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n \left( t \right)_n q^{n(n-3)+n\alpha z^n \left( \frac{q}{q^2} \right)^n}}{(-\frac{z^2}{q}; q)_{2n+1}} + \frac{z}{q} \prod_{n=0}^{N} \frac{(-1)^n \left( t \right)_n q^{n(2n-1)} + \alpha z^n \left( \frac{q}{q^2} \right)^n}{(\frac{z^2}{q}; q)_{2n+1}}$

$= \frac{1}{(t)_\infty} \sum_{n=0}^{N} \frac{(-1)^n \left( t \right)_n q^{n(n-3)+n\alpha z^n \left( \frac{q}{q^2} \right)^n}}{(-\frac{z^2}{q}; q)_{2n+1}} + \frac{z}{q} \psi_{0,N}(t, \alpha, z; q)$

Which proves Theorem 2 (i).

Proof of (ii)

$\rho_{0,N}(t, \alpha, z; q) = \frac{z(1 + \frac{z}{q})}{2(t)_\infty} \sum_{n=0}^{N} \frac{\left( t \right)_n q^{n(n-1)+n\alpha z^n \left( \frac{q}{q^2} \right)^n}}{(\frac{z^2}{q}; q^2)_{n+1}}$

$= \frac{z}{2} \left( 1 + \frac{z}{q} \right) D_{q,t} \rho_{0,N}(t, \alpha, z; q)$.

Which proves Theorem 2 (ii).

Proof of (iii)

Writing $\alpha^2$ for $\alpha$ in $\phi_{0,N}(t, \alpha, \beta, z; q)$, we have

$D_{q,t} \phi_{0,N}(t, \alpha^2, \beta, z; q) = \frac{1}{\left( t \right)_\infty} \sum_{n=0}^{N} \frac{\left( t \right)_n q^{n^2}-2n^2+n\beta z^n \left( \frac{q}{q^2} \right)^n}{(\frac{-\alpha^2 z^2}{q}; q^2)_{n+1}} = (1 + \frac{\alpha^2 z^2}{q^2}) \nu_{0,N}(t, \alpha, \beta, z; q)$

Which proves Theorem 2 (iii).

Proof of (iv)

Writing $\frac{-\alpha}{q}$ for $\alpha$ and then $\alpha^2$ for $\alpha$ in $\psi_{0,N}(t, \alpha, \beta, z; q)$, we have

$\psi_{0,N}(t, \alpha^2, \beta, z; q) = \frac{z}{\left( t \right)_\infty} \sum_{n=0}^{N} \frac{\left( t \right)_n q^{n^2}-2n^2+n\beta z^n \left( \frac{q}{q^2} \right)^n}{(\frac{-\alpha^2 z^2}{q}; q^2)_{n+1}} = z D_{q,t} \nu_{0,N}(t, \alpha, \beta, z; q)$

which proves Theorem 2(iv).
5 q-Integral Representation for the Generalized Partial Sixth Order Mock Theta Function and Generalized Partial Third Order Mock Theta Functions:

The q-integral was defined by Thomas and Jackson[7,p.19] as

\[
\int_0^1 f(t) d_q t = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n
\]

We now give the q-integral representation for the generalized sixth order mock theta functions and also for generalized third order mock theta functions.

**Theorem 3(a)**

(i) \(\phi_{0,N}(q^t, \alpha, z; q) = \frac{(1 - q)^{-1}}{(q; q)_{\infty}} \int_0^1 t^{t-1} (\omega q; q)_{\infty} \phi_{0,N}(0, a\omega, z; q) d_q \omega\).

(ii) \(\psi_{0,N}(q^t, \alpha, z; q) = \frac{(1 - q)^{-1}}{(q; q)_{\infty}} \int_0^1 t^{t-1} (\omega q; q)_{\infty} \psi_{0,N}(0, a\omega, z; q) d_q \omega\).

(iii) \(\rho_{0,N}(q^t, \alpha, z; q) = \frac{(1 - q)^{-1}}{(q; q)_{\infty}} \int_0^1 t^{t-1} (\omega q; q)_{\infty} \rho_{0,N}(0, a\omega, z; q) d_q \omega\).

(iv) \(\gamma_{0,N}(q^t, \alpha, z; q) = \frac{(1 - q)^{-1}}{(q; q)_{\infty}} \int_0^1 t^{t-1} (\omega q; q)_{\infty} \gamma_{0,N}(0, a\omega, z; q) d_q \omega\).

(v) \(\sigma_{0,N}(q^t, \alpha, z; q) = \frac{(1 - q)^{-1}}{(q; q)_{\infty}} \int_0^1 t^{t-1} (\omega q; q)_{\infty} \sigma_{0,N}(0, a\omega, z; q) d_q \omega\).

**Proof**

We shall give the detailed proof for \(\phi_{0,N}(q^t, \alpha, z; q)\). The proof for the other functions are similar, so omitted.

**Limiting case of q-beta integral [7,p.19(1.11.7)] is**

\[
\frac{1}{(q^\omega; q)_{\infty}} = \frac{(1 - q)^{-1}}{(q; q)_{\infty}} \int_0^1 t^{t-1} (tq; q)_{\infty} d_q t.
\]

Now

\[
\phi_{0,N}(t, \alpha, z; q) = \frac{1}{(t)_{\infty}} \sum_{n=0}^{N} \frac{(-1)^n(t)_{n} q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_{n}}{(-\frac{z^2}{q}; q)_{2n}}.
\]

Replacing \(t\) by \(q^t\) and \(q^\alpha\) by \(a\), we have

\[
\phi_{0,N}(q^t, \alpha, z; q) = \frac{1}{(q^t)_{\infty}} \sum_{n=0}^{N} \frac{(-1)^n(q^t)_{n} q^{n(n-3)+n\alpha} z^{2n} (\frac{z^2}{q}; q^2)_{n}}{(-\frac{z^2}{q}; q)_{2n}}.
\]
But by using (5.1), (5.2) can be written as

\[ \sum_{n=0}^{N} (-1)^n q^{n^3 + n \alpha} z^{2n}(\frac{z^2}{q^{2n}}) (1-q)^{-1} \int_0^1 \omega^{n+t-1}(\omega q; q) \infty dq \omega, \]

\[ = \frac{(1-q)^{-1}}{(q; q) \infty} \int_0^1 \omega^{t-1}(\omega q; q) \infty \sum_{n=0}^{N} (-1)^n q^{n^3 + n \alpha} z^{2n}(\frac{z^2}{q^{2n}}) (a \omega)^n dq \omega \]

(5.1)

But

\[ \phi_{0,N}(0, \alpha, z; q) = \sum_{n=0}^{N} (-1)^n q^{n^3 + n \alpha} z^{2n}(\frac{z^2}{q^{2n}}) (\frac{z^2}{\alpha}) \]

and since \( q^n = a \),

\[ \phi_{0,N}(0, a, z; q) = \sum_{n=0}^{N} (-1)^n a q^{n^3 + n \alpha} z^{2n}(\frac{z^2}{q^{2n}}), \]

Hence

\[ \phi_{0,N}(0, a \omega, z; q) = \sum_{n=0}^{N} (-1)^n (a \omega) q^{n^3 + n \alpha} z^{2n}(\frac{z^2}{q^{2n}}), \]

(5.2)

By using (5.1), (5.2) can be written as

\[ \phi_{0,N}(q', \alpha, z; q) = \frac{(1-q)^{-1}}{(q; q) \infty} \int_0^1 \omega^{t-1}(\omega q; q) \infty \phi_{0,N}(0, a \omega, z; q) dq \omega. \]

which proves (i). The proofs for all other functions are similar.

**Theorem 3(b):**

The q-integral representation for the generalized partial third order mock theta functions:

(i) \( f_{0,N}(q', \alpha, \beta, z; q) = \frac{(1-q)^{-1}}{(q; q) \infty} \int_0^1 \omega^{t-1}(\omega q; q) \infty f_{0,N}(0, a \omega, z; q) dq \omega. \)

(ii) \( \phi_{0,N}(q', \alpha, \beta, z; q) = \frac{(1-q)^{-1}}{(q; q) \infty} \int_0^1 \omega^{t-1}(\omega q; q) \infty \phi_{0,N}(0, a \omega, z; q) dq \omega. \)

(iii) \( \psi_{0,N}(q', \alpha, \beta, z; q) = \frac{(1-q)^{-1}}{(q; q) \infty} \int_0^1 \omega^{t-1}(\omega q; q) \infty \psi_{0,N}(0, a \omega, z; q) dq \omega. \)

(iv) \( \nu_{0,N}(q', \alpha, \beta, z; q) = \frac{(1-q)^{-1}}{(q; q) \infty} \int_0^1 \omega^{t-1}(\omega q; q) \infty \nu_{0,N}(0, a \omega, z; q) dq \omega. \)

(v) \( \chi_{0,N}(q', \alpha, \beta, z; q) = \frac{(1-q)^{-1}}{(q; q) \infty} \int_0^1 \omega^{t-1}(\omega q; q) \infty \chi_{0,N}(0, a \omega, z; q) dq \omega. \)

(vi) \( \rho_{0,N}(q', \alpha, \beta, z; q) = \frac{(1-q)^{-1}}{(q; q) \infty} \int_0^1 \omega^{t-1}(\omega q; q) \infty \rho_{0,N}(0, a \omega, z; q) dq \omega. \)

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\[(vii)\omega_{0,N}(q^t, \alpha, \beta, z; q) = \frac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 \omega^{-1}(\omega q; q)_\infty \omega_{0,N}(0, a, \omega, z; q) d_q \omega.\]

**Proof:**
The proofs are similar to given above for \(\phi_{0,N}(q^t, \alpha, z; q)\), so the Theorem 3(b) follows.

**Conclusions:** Mock theta functions are mysterious functions. These investigations will be helpful in understanding more about these partial mock theta functions. Being shown that they belong to the class of \(F_q\)-functions and properties are established for the partial mock theta functions and relations between these partial mock theta functions can also be derived.

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**References**


