Inexact Kleinman-Newton-ADI Method with Line Search to Solve Large-Scale Algebraic Riccati Equations

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Algebraic Riccati Equation

$$\mathcal{R}(X) = C^T C + A^T X + XA - XBB^T X = 0$$

Inexact Kleinman-Newton and Line Search

Specially tailored Newton method $\Rightarrow$ **Kleinman-Newton method**:

$$\mathcal{R}'(X^{(k)})X^{(k+1)} = \mathcal{R}'(X^{(k)})X^{(k)} - \mathcal{R}(X^{(k)}).$$

$\Rightarrow$ Converges globally for any stabilizing $X^{(0)} \in \mathbb{R}^{n \times n}$.

Problems:

- often $\|\mathcal{R}(X^{(1)})\| \gg \|\mathcal{R}(X^{(0)})\|$
 $\Rightarrow$ Apply **line search** idea [1]; **BUT**: involves dense computations.

- Newton step computed iteratively (necessary accuracy unknown).
 $\Rightarrow$ Apply **inexact Kleinman-Newton** idea [4] and solve

$$(A^{(k)})^T X^{(k+1)} + X^{(k+1)} A^{(k)} = -G^{(k)}(G^{(k)})^T + L^{(k+1)}.$$

BUT: How to monitor residual $L^{(k+1)}$ and check conv. conditions?

Low-Rank Residual ADI [3]

Newton step k :

$$(A^{(k)})^T X + XA^{(k)} = -G^{(k)}(G^{(k)})^T,$$

with $A^{(k)} = A - BB^T X^{(k)}$ and

$G^{(k)} = [C^T \mid X^{(k)}B]$ until $\|L^{(k+1)}\| \leq \text{tol}_{ADI}$.

$\Rightarrow$ Apply **low-rank residual ADI** method [3].

Advantages:

- Lyapunov residual has low-rank structure:
 $L^{(k+1)} = WW^T$, $W \in \mathbb{R}^{n \times p}$, $p \ll n$,
 $\text{rank}(L^{(k+1)}) = \text{rank}(W) \leq \text{rank}(G) \ll n$.
- Low-rank structure is key for efficiency.

Algorithm:

INPUT: A , B , C , and stabilizing $X^{(0)}$

OUTPUT: stabilizing spsd solution $X^{(*)}$

1. Set $k = 0$ and $\bar{\eta} \in (0, 1)$.
2. **while** $\|\mathcal{R}(X^{(k)})\| > \text{tol}_{\text{Newton}}$ **do**
3. $A^{(k)} = A - BB^T X^{(k)}$, $G = [C^T \mid X^{(k)}B]$
4. Select $\eta_k \leq \bar{\eta}$ and $\alpha \in (0, 1 - \bar{\eta})$.
5. Compute spsd $\tilde{X}$ such that
 $(A^{(k)})^T \tilde{X} + \tilde{X} A^{(k)} = -GG^T + L^{(k+1)}$,
with $\|L^{(k+1)}\| \leq \eta_k \|\mathcal{R}(X^{(k)})\|$.
6. Compute **step size** $\lambda_k > 0$ such that
 $\|\mathcal{R}((1 - \lambda_k)X^{(k)} + \tilde{X})\| \leq (1 - \lambda_k \alpha) \|\mathcal{R}(X^{(k)})\|$.
7. Set $X^{(k+1)} = (1 - \lambda_k)X^{(k)} + \lambda_k \tilde{X}^{(k)}$
8. $k = k + 1$
9. **end while**
10. $X^{(k)} = X^{(*)}$

Key ingredients:

- Riccati residual as symmetric indefinite low-rank product

$$\mathcal{R}(X) = UDU^T \in \mathbb{R}^{n \times n}$$

with $U = [W \mid \Delta K] \in \mathbb{R}^{n \times (p+q)}$, $D = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$,
and $p + q \ll n$.

- Line search for large-scale case.
 $\Rightarrow$ Exploits low-rank structure of residuals.
 $\Rightarrow$ Highly efficient computations only use "small" inner products of low-rank factors.
 $\Rightarrow$ Direct monitoring of $\|\mathcal{R}(X)\|$ and $\|L\|$.

Convergence results:

- Convergence proof in [4] is **not feasible** in the low-rank case.
 - Use inexact Kleinman-Newton-ADI with line search and the following **assumptions**:
 - (A, B) stabilizable.
 - (C, A) detectable.
 - λ_k is bounded away from 0.
 - $X^{(k)}$ spsd & stabilizing $\forall k \geq K > 0$.
- $\Rightarrow X^{(k)} \rightarrow X^{(*)}$, the unique stabilizing solution.

2D Heat Equation Example ($n = 529$, $p = 26$, $q = 25$)

References

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