STRUCTURED EIGENVALUE BACKWARD ERRORS OF MATRIX POLYNOMIALS

(VM60 Conference, May 2015, TU Berlin)

Shreemayee Bora[†], Michael Karow[‡], Christian Mehl[‡] and Punit Sharma[†]

[†]Department of Mathematics, IIT Guwahati, Assam, India and [‡]Institut für Mathematik, TU Berlin, Berlin, Germany

Matrix polynomials and their structures

Let $(A_0, \dots, A_m) \in \mathbb{S} \subset (\mathbb{C}^{n \times n})^{m+1}$ and $\star \in \{*, T\}$.

Matrix polynomial : $P(z) = \sum_{i=0}^m z^i A_i$

Structured matrix polynomials

Structure	Coefficient Properties
Hermitian	$A_i^* = A_i$
Symmetric	$A_i^T = A_i$
$\star$ -even	$A_i = (-1)^i A_i^*$
$\star$ -odd	$A_i = (-1)^{i+1} A_i^*$
$\star$ -palindromic	$A_i = A_{m-i}^*$
$\star$ -antipalindromic	$A_i = -A_{m-i}^*$

Applications

1. In control theory,
2. In vibration analysis,
3. In the numerical solutions of higher order system of ordinary and partial differential equations, etc.

(The Millennium footbridge problem)

In this problem the modes and frequencies of the structure correspond to the solutions of a *quadratic polynomial* $Q(\lambda) = \lambda^2 M + \lambda C + K$ when damping effects are induced.

(Vibration analysis of rail tracks excited by high speed trains)

Finite element discretization of rail leads to a *palindromic eigenvalue problem* $(\lambda^2 A_0^T + \lambda A_1 + A_0)x = 0$, where $A_0, A_1 \in \mathbb{C}^{n \times n}$, and $A_1^T = A_1$.

Objective: Structured eigenvalue backward errors

Objective: Let $P(z) = \sum_{i=0}^m z^i A_i$, where $(A_0, \dots, A_m) \in \mathbb{S}$ and $\lambda \in \mathbb{C}$. Then calculate

$$\eta_{\infty}^{\mathbb{S}}(P, \lambda) = \inf \left\{ \max \{ \|\Delta_0\|, \dots, \|\Delta_m\| \} \mid \Delta = (\Delta_0, \dots, \Delta_m) \in \mathbb{S}, \Delta P(z) = \sum_{j=0}^m z^j \Delta_j, \det(P(\lambda) - \Delta P(\lambda)) = 0 \right\}.$$

- When $\mathbb{S} = (\mathbb{C}^{n \times n})^{m+1}$, then $\eta_{\infty}(P, \lambda) := \eta_{\infty}^{\mathbb{S}}(P, \lambda)$ is called *unstructured backward error*.
- When $\mathbb{S} = (\mathbb{R}^{n \times n})^{m+1}$, then $\eta_{\infty}^{\mathbb{R}}(P, \lambda)$ is called the real eigenvalue backward error.
- **Explicit formulae for $\eta_{\infty}^{\mathbb{S}}(P, \lambda)$ are not known in literature even for $m = 1$ case.**

Hermitian matrix polynomials

Theorem 1: Let $P(z)$ be a Hermitian polynomial and $\lambda \in \mathbb{C} \setminus \{0\}$ be such that $M = (P(\lambda))^{-1}$ exists. Then

- (1) There exists $\gamma^* = (\gamma_0^*, \dots, \gamma_m^*)$ where γ_i are all nonzero and Hermitian matrices $G, H_0, \dots, H_m$ depending on γ^* such that

$$\lambda_{\max}^* := \inf_{t_0, \dots, t_m \in \mathbb{R}} \lambda_{\max}(G + t_0 H_0 + \dots + t_m H_m)$$

is attained for some $t_0^*, \dots, t_m^* \in \mathbb{R}$. If $\lambda_{\max}^*$ is simple, then

$$\eta_{\infty}(P, \lambda) \leq \frac{1}{\sqrt{\lambda_{\max}^*}} \leq \eta_{\infty}^{\text{Herm}}(P, \lambda) \leq \sqrt{(m+1)} \frac{1}{\sqrt{\lambda_{\max}^*}}.$$

- (2) Under certain conditions which are observed to be satisfied numerically, we have

$$\eta_{\infty}^{\text{Herm}}(P, \lambda) = \frac{1}{\sqrt{\lambda_{\max}^*}}.$$

Further, if polynomial is real then

$$\eta_{\infty}^{\text{Herm}_{\mathbb{R}}}(P, \lambda) = \eta_{\infty}^{\text{Herm}}(P, \lambda),$$

where $\eta_{\infty}^{\text{Herm}_{\mathbb{R}}}(P, \lambda)$ is the real Hermitian eigenvalue backward error preserving the real as well as the Hermitian structure of $P(z)$.

Numerical results

Example 1:

For a random Hermitian pencil $L(z) = A_0 + zA_1$ of size 4×4 with eigenvalue $0.5766 \pm 1.0199i, -1.0966$ and -0.1019 .

$$\lambda = -1.0966 + 0.5i$$

$$\gamma_1 = 0.6946 \text{ and } \gamma_2 = 0.7194$$

$$\eta_{\infty}(L, \lambda) = 0.3341 \text{ and } \eta_{\infty}^{\text{Herm}}(L, \lambda) = 0.9238.$$

Figure 1. Eigenvalue perturbation curves for the Hermitian pencil $L(z)$ of Example 1 with respect to Hermitian perturbations (left) and arbitrary perturbations (right).

Example 2:

For a random Hermitian quadratic polynomial $Q(z) = A_0 + zA_1 + z^2A_2$ of size 3×3 with eigenvalue $-0.8738 \pm 2.4984i, 0.3091 \pm 1.2260i$ and 0.6280 and 0.0780

$$\lambda = 0.6280 + 0.5i$$

$$\gamma_1 = 0.6823, \gamma_2 = 0.5249 \text{ and } \gamma_3 = 0.5090.$$

$$\eta_{\infty}(Q, \lambda) = 0.7788 \text{ and } \eta_{\infty}^{\text{Herm}}(Q, \lambda) = 1.1622.$$

Figure 2. Eigenvalue perturbation curves for the Hermitian quadratic polynomial $Q(z)$ of Example 2 with respect to Hermitian perturbations (left) and arbitrary perturbations (right).

Real eigenvalue backward errors of matrix pencils

Theorem 2. Let $L(z) = A + zB$, $A, B \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ with $M = (L(\lambda))^{-1}$. For $\gamma > 0$ and $\delta \in \mathbb{C} \setminus \{0\}$, define

$$R(\gamma, \delta) := \begin{bmatrix} \text{Re } M_{\delta} & -\gamma \text{Im } M_{\delta} \\ \gamma^{-1} \text{Im } M_{\delta} & \text{Re } M_{\delta} \end{bmatrix} \text{ where } M_{\delta} := \begin{bmatrix} M & \delta \lambda M \\ \delta^{-1} M & \lambda M \end{bmatrix}. \text{ Then}$$

$$\eta_{\infty}(L, \lambda) \leq \underbrace{\left(\inf_{\gamma \in (0,1]} \inf_{\delta \in \mathbb{C} \setminus \{0\}} \sigma_2(R(\gamma, \delta)) \right)^{-1}}_{=: \text{lbound}} \leq \eta_{\infty}^{\mathbb{R}}(L, \lambda) \leq \min \{ \eta_{\infty, A}^{\mathbb{R}}, \eta_{\infty, B}^{\mathbb{R}} \},$$

where $\eta_{\infty, A}^{\mathbb{R}}$ and $\eta_{\infty, B}^{\mathbb{R}}$ are the real eigenvalue backward errors with the perturbation in only A and B , respectively, and $\sigma_2(R(\gamma, \delta))$ is the second largest singular value of $R(\gamma, \delta)$.

Further, under certain conditions we have

$$\eta_{\infty}^{\mathbb{R}}(L, \lambda) = \left(\min_{\gamma \in (0,1]} \min_{\delta \in \mathbb{C} \setminus \{0\}} \sigma_2(R(\gamma, \delta)) \right)^{-1}.$$

Numerical Results

Table 1

Tightness of lower bound for arbitrarily chosen Hermitian pencils $L(z)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here lbound is the lower bound of $\eta_{\infty}^{\mathbb{R}}(L, \lambda)$ obtained in Theorem-2 and since $L(z)$ is Hermitian $\eta_{\infty}^{\text{Herm}}(L, \lambda)$ gives an upper bound of $\eta_{\infty}^{\mathbb{R}}(L, \lambda)$

size	$\eta_{\infty}(L, \lambda)$	lbound	$\eta_{\infty}^{\text{Herm}}(L, \lambda)$
2x2	0.4929	0.6056	0.6059
3x3	0.2909	0.3394	0.3741
4x4	0.5205	0.8181	0.8646
5x5	0.3135	0.3475	0.3611
6x6	0.5156	0.5753	0.6046

Figure 3

Figure 3. A random real Hermitian pencil of size 3×3 with an eigenvalue 2.7922 showing that $\eta_{\infty}(L, \lambda) \rightarrow 0$ as $\lambda \rightarrow 2.7922$ but $\eta_{\infty}^{\mathbb{R}}(L, \lambda) \rightarrow 0$ as $\lambda \rightarrow 2.7922$.

Table 2

size	$\eta_{\infty}(L, \lambda)$	$\eta_{\infty}^{\mathbb{R}}(L, \lambda)$
2x2	0.8801	0.9243
3x3	0.4498	0.4638
4x4	0.4913	0.5320
5x5	0.6085	0.6132
6x6	0.4986	0.8227

Table 2. $\eta_{\infty}(L, \lambda)$ and $\eta_{\infty}^{\mathbb{R}}(L, \lambda)$ for random pencils and $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Table 3

size	$\eta_{\infty}(L, \lambda)$	$\eta_{\infty}^{\mathbb{R}}(L, \lambda)$	$\eta_{\infty}^{\text{Herm}}(L, \lambda)$
2x2	0.4093	0.5301	0.5301
3x3	1.1851	1.2479	1.2479
4x4	0.4637	0.6708	0.6708
5x5	0.4425	0.5152	0.5152
6x6	1.0419	1.1466	1.1466

Table 3. $\eta_{\infty}(L, \lambda)$ and $\eta_{\infty}^{\mathbb{R}}(L, \lambda)$ for random Hermitian pencils and $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Real structured eigenvalue backward errors

- (1) Let $L(z) = A + zB$, where $A, B \in \mathbb{R}^{n \times n}$ with $A^T = A$ and $B^T = -B$. Then
$$\eta_{\infty}(L, \lambda) \leq \eta_{\infty}^{\text{Even}_{\mathbb{R}}}(L, \lambda) \leq (1 + |\lambda|) \eta_{\infty}(L, \lambda), \text{ if } \lambda = i\theta, \quad (1)$$

where $\eta_{\infty}^{\text{Even}_{\mathbb{R}}}(L, \lambda)$ denotes the real even eigenvalue backward error.

- (2) Let $L(z) = A + zA^T$, where $A \in \mathbb{R}^{n \times n}$. Then

$$\eta_{\infty}(L, \lambda) \leq \eta_{\infty}^{\text{Pal}_{\mathbb{R}}}(L, \lambda) \leq \sqrt{1 + \tan^2 \left(\frac{\theta}{2} \right)} \eta_{\infty}(L, \lambda), \text{ if } \lambda = e^{i\theta}, \quad (2)$$

where $\eta_{\infty}^{\text{Pal}_{\mathbb{R}}}(L, \lambda)$ denotes the real palindromic eigenvalue backward error.

Figure 4. Tightness of bounds on $\eta_{\infty}^{\text{Even}_{\mathbb{R}}}(L, \lambda)$ given by (1) for $\lambda = ix$ where $-1 < x < 1$.

Figure 5. Tightness of bounds on $\eta_{\infty}^{\text{Pal}_{\mathbb{R}}}(L, \lambda)$ given by (2) for $\lambda = e^{ix}$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Remark: Similar kind of bounds can also be obtained for real odd and real antipalindromic pencils for λ values chosen on imaginary axis and unit circle, respectively.

Acknowledgement: This work was supported by DST, India and DAAD, Germany through the project *Distance Problems for Structured Matrix Polynomials* within the Personal Exchange Programme PPP.

Publications:

- [1] S. Bora, M. Karow, C. Mehl, and P. Sharma. Structured Eigenvalue Backward Errors of Matrix Pencils and Polynomials with Hermitian and Related Structures. *SIAM J. Matrix Anal. Appl.*, 35:2, 453-475, 2014.
- [2] S. Bora, M. Karow, C. Mehl, and P. Sharma. Structured Eigenvalue Backward Errors of Matrix Pencils and Polynomials with Palindromic Structures. *SIAM J. Matrix Anal. Appl.*, 36:2, 393-416, 2015.