Stochastic Galerkin reduced basis methods for parametrized random elliptic PDEs

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We consider the estimation of parameter-dependent statistical outputs for parametrized elliptic PDE problems with random data. We propose a stochastic Galerkin reduced basis (SGRB) method, which provides the expected output for a given parameter value at the cost of solving a *single* low-dimensional system of equations. This is substantially faster than usual Monte Carlo reduced basis (MCRB) methods, which require *multiple* samples of the reduced solution.

Notation

	Monte Carlo	stochastic Galerkin
original model	$u: \Xi^M \times P \rightarrow X$	$u_{\text{SG}}: P \rightarrow S \times X$
snapshots	$u: \Xi^M \times P^N \rightarrow X$	$u_{\text{SG}}: P^N \rightarrow S \times X$
reduced space	$X_R \subset X$	$(S \times X)_R \subset S \times X$
reduced model	$u^R: \Xi^M \times P \rightarrow X_R$	$u_{\text{SG}}^R: P \rightarrow (S \times X)_R$

P	deterministic parameter domain	$P^N \subset P$	set of Monte Carlo samples $\mu^1, \dots, \mu^N$
Ξ	random parameter domain	$\Xi^M \subset \Xi$	set of Monte Carlo samples $y^1, \dots, y^M$
X	finite element (FE) space	$S \subset L^2(\Xi)$	stochastic Galerkin (SG) space

Monte Carlo reduced basis method [1, 2, 3]

Problem 1 Given $(y, \mu) \in \Xi \times P$, find

$$\begin{aligned} u(y, \mu) \in H: \quad & a(u(y, \mu), v; y, \mu) = f(v; y, \mu) \quad \forall v \in H, \\ s(y, \mu) \in \mathbb{R}: \quad & s(y, \mu) = l(u(y, \mu); y, \mu). \end{aligned}$$

original model	Obtain $u(y, \mu)$, $s(y, \mu)$ from Problem 1 with $H = X$.
snapshots	Solutions $u(y^1, \mu^1), \dots, u(y^M, \mu^N) \in X$ of original model.
reduced space	$X_R = \text{span}(\phi^1, \dots, \phi^R) \subset \text{span}(u(y^1, \mu^1), \dots, u(y^M, \mu^N))$.
reduced model	Obtain $u^R(y, \mu)$, $s^R(y, \mu)$ from Problem 1 with $H = X_R$.

- Monte Carlo reduced basis estimate of $\mathbb{E}[s(\cdot, \mu)] := \int_{\Xi} s(y, \mu) \rho(y) dy$:

$$\mathbb{E}[s(\cdot, \mu)] \stackrel{\text{MC}}{\approx} E_{\text{MC}}[s(\cdot, \mu)] \stackrel{\text{RB}}{\approx} E_{\text{MC}}[s^R(\cdot, \mu)], \quad \text{where } E_{\text{MC}}[s(\cdot, \mu)] := \frac{1}{M} \sum_{m=1}^M s(y^m, \mu).$$

- A posteriori error bound:

$$\left| E_{\text{MC}}[s(\cdot, \mu)] - E_{\text{MC}}[s^R(\cdot, \mu)] \right| \leq E_{\text{MC}} \left[\frac{\|l(\cdot; \cdot, \mu)\|_{X'} \|f(\cdot; \cdot, \mu) - a(u^R(\cdot, \mu), \cdot; \cdot, \mu)\|_{X'}}{\inf_{v \in X} a(v, v; \cdot, \mu) / \|v\|_X^2} \right].$$

Stochastic Galerkin reduced basis method [4, 5]

Problem 2 Given $\mu \in P$, find

$$\begin{aligned} u(\mu) \in H: \quad & \mathbb{E}[a(u(\mu), v; \cdot, \mu)] = \mathbb{E}[f(v; \cdot, \mu)] \quad \forall v \in H, \\ s(\mu) \in \mathbb{R}: \quad & s(\mu) = \mathbb{E}[l(u(\mu); \cdot, \mu)]. \end{aligned}$$

original model	Obtain $u_{\text{SG}}(\mu)$, $s_{\text{SG}}(\mu)$ from Problem 2 with $H = S \times X$.
snapshots	Solutions $u_{\text{SG}}(\mu^1), \dots, u_{\text{SG}}(\mu^N) \in S \times X$ of original model.
reduced space	$(S \times X)_R = \text{span}(\phi_{\text{SG}}^1, \dots, \phi_{\text{SG}}^R) \subset \text{span}(u_{\text{SG}}(\mu^1), \dots, u_{\text{SG}}(\mu^N))$.
reduced model	Obtain $u_{\text{SG}}^R(\mu)$, $s_{\text{SG}}^R(\mu)$ from Problem 2 with $H = (S \times X)_R$.

- Stochastic Galerkin reduced basis estimate of $\mathbb{E}[s(\cdot, \mu)] := \int_{\Xi} s(y, \mu) \rho(y) dy$:

$$\mathbb{E}[s(\cdot, \mu)] \stackrel{\text{SG}}{\approx} s_{\text{SG}}(\mu) \stackrel{\text{RB}}{\approx} s_{\text{SG}}^R(\mu).$$

- A posteriori error bound:

$$\left| s_{\text{SG}}(\mu) - s_{\text{SG}}^R(\mu) \right| \leq \frac{\left\| \mathbb{E}[l(\cdot; \cdot, \mu)] \right\|_{(S \times X)'} \left\| \mathbb{E}[f(\cdot; \cdot, \mu)] - \mathbb{E}[a(u_{\text{SG}}^R(\mu), \cdot; \cdot, \mu)] \right\|_{(S \times X)'} }{\inf_{v \in S \times X} \mathbb{E}[a(v, v; \cdot, \mu)] / \|v\|_{S \times X}^2}.$$

Discussion

Offline cost (snapshot computation):

- MCRB**: Solve MN FE systems.
- SGRB**: Solve N block-structured SGFE systems.
- Decouple SGFE systems using double-orthogonal tensor product polynomials.
- Rigorous error bound allows further approximation of snapshots (ongoing work).

Online cost (parameter-dependent statistical estimation):

- MCRB**: Solve M reduced systems per parameter sample $\mu \in P$,
- SGRB**: Solve 1 reduced system per parameter sample $\mu \in P$.

Primal/dual formulation:

- Improve reduced basis outputs and error bounds via additional dual problems.

Example problem

- Steady convection-diffusion-reaction in a unit square:

$$\begin{aligned} \mu \cdot \nabla u(y, \mu) - \Delta u(y, \mu) - \alpha(y)u &= 1, & x \in \Omega, \\ u(y, \mu) &= 0, & x \in \partial\Omega, \\ l(u(y, \mu); y, \mu) &= \int_{\Omega} u(y, \mu) dx. \end{aligned}$$

- Deterministic convective velocity parameter $\mu \in 1000 \cdot [-1, 1]^2$.
- Random reactivity field $\alpha(y)$ via KL-decomposition: $y \in 100 \cdot [-\sqrt{3}, \sqrt{3}]^8$.

original model	Ξ^M Monte Carlo with $M = 1024$ random samples of $y \in \Xi$. S Stochastic Galerkin with 256 degrees of freedom. X Finite elements with 225 degrees of freedom.
snapshots	P^N Monte Carlo with $N = 64$ random samples of $\mu \in P$.
reduced space	Proper orthogonal decomposition based on H -inner product.
reduced model	$N_{\text{test}} = 64$ samples for $L^2(P)$ -norm of error in expected output.

- Realizations of random reactivity:

- Realizations of solution and output for fixed convection:

- Realizations of solution and output for fixed reactivity:

Results

Conclusions

- Error bounds certify accuracy with respect to original SGFE and MCFE discretizations.
- We observe similar convergence of SGRB and MCRB depending on reduced dimension.
- The online phase of SGRB is $O(M)$ faster than MCRB, where M is the number of MC samples.

References

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Acknowledgment

The work of Sebastian Ullmann is supported by the Excellence Initiative of the German federal and state governments and the Graduate School of Computational Engineering at the Technische Universität Darmstadt.