Adaptive POD-DEIM model reduction based on an improved error estimator

Sridhar Chellappa¹, Lihong Feng¹, Peter Benner¹

¹Max Planck Institute for Dynamics of Complex Technical Systems, Computational Methods in Systems and Control Theory, Sandtorstraße 1, Magdeburg, Germany.

Model reduction for nonlinear dynamical systems

For reliable, efficient, rapid simulations of dynamical systems, a reduced order model (ROM) with certified accuracy is highly desirable. The ROM is derived from a full order model (FOM) through model reduction. In this work we propose an adaptive approach for nonlinear model reduction by making use of a suitable output error estimator, thus enable the generation of a compact ROM, with appropriate balance between the state and nonlinear approximations.

$$\begin{aligned} \dot{x} &= A x + g(x) + B u \\ y &= C x \end{aligned} \quad \xrightarrow{x \approx V x_r} \quad \begin{aligned} \dot{x}_r &= A_r x_r + g_r(x_r) + B_r u \\ y_r &= C_r x_r \end{aligned}$$

- $x \in \mathbb{R}^n$ - state vector,
- $u \in \mathbb{R}^m$ - input vector,
- $y \in \mathbb{R}^p$ - output vector,
- $g(\cdot) \in \mathbb{R}^n$ - nonlinear term,
- $\hat{g}(\cdot) \in \mathbb{R}^n$ - approximated nonlinear term,
- $A \in \mathbb{R}^{n \times n}$ - state matrix,
- $B \in \mathbb{R}^{n \times m}$ - input matrix,
- $C \in \mathbb{R}^{p \times n}$ - output matrix,
- $V \in \mathbb{R}^{n \times r}$ - projection matrix,
- $A_r, B_r, C_r, x_r, y_r, g_r$ - reduced quantities and $g_r(\cdot) := V^T \hat{g}(\cdot)$.

Snapshots-based model reduction

Proper Orthogonal Decomposition (POD)

$$\text{Snapshots matrix } X = [x(t_1) x(t_2) \dots x(t_k)] \xrightarrow{SVD} M \Sigma R^T$$

- Projection matrix $V = M(:, 1:r)$. r chosen from the decay of singular values such that,
- $\frac{\sum_{i=r+1}^n \sigma_i}{\sum_{i=1}^r \sigma_i} < \text{tolerance}$

Drawback: If only some output quantity is of interest, such an approach is very conservative. Also, it is not possible, a priori, to determine the value of r , such that the ROM respects some predefined tolerance.

Remedy: An adaptive approach, making use of an appropriate error estimator (see next section).

Discrete Empirical Interpolation Method (DEIM)

- POD leads to dimension reduction but no complexity reduction for nonlinear term.
- [1] proposes a method to reduce the complexity.

$$\text{Nonlinear term snapshots matrix } G = [g(x(t_1)) g(x(t_2)) \dots g(x(t_k))] \xrightarrow{SVD} J \Sigma_{deim} L^T$$

- DEIM projection matrix $U_{deim} = J(:, 1:\ell)$, $\hat{g}(x) \approx U_{deim} C$
- ℓ chosen the same way as done for r in the POD case above.

A posteriori error estimation

Choose r, ℓ adaptively using an *a posteriori* error estimator.

- Discretization: Finite volume/Finite difference method in space and Semi-implicit Euler method in time.
- Primal-dual formulation for improved convergence and effectivity [2, 3, 4].

$$\text{Primal system: } E_d x^{k+1} = A_d x^k + \Delta t \hat{g}(x^k) + \Delta t B u^k \quad \text{Dual system: } E_d^T x_{du}^{k+1} = -C^T$$

Error estimator

$$\Delta := \|y^{k+1} - \hat{y}^{k+1}\| \leq \phi^{k+1} \|r_{pr}^{k+1}\|$$

$$\text{where, } \phi^{k+1} := S^{k+1} \left(\|E_d^{-1}\| \|r_{du}\| + \|\hat{x}_{du}\| \right), \quad S^{k+1} := \frac{\|\tilde{r}^{k+1}\|}{\|r^{k+1}\|}$$

Improved error estimator

To achieve better effectivity, we consider the corrected output given as, $\tilde{y} = \hat{y} - \hat{x}_{du}^T r_{pr}$

$$\text{where, } \tilde{\Delta} := \|y^{k+1} - \tilde{y}^{k+1}\| \leq \omega^{k+1} \|r_{pr}^{k+1}\|$$

$$\omega^{k+1} := S^{k+1} \|E_d^{-1}\| \|r_{du}\| + |1 - S^{k+1}| \|\hat{x}_{du}\|$$

Note: The primal system residual can be split to give the POD and DEIM error contributions separately, i.e. $r_{pr}^{k+1} = r_{DEIM,pr}^{k+1} + e_{DEIM}^{k+1}$. Therefore, the error estimator itself can be separated into POD and DEIM contributions as, $\Delta_{POD}, \Delta_{DEIM}$.

Adaptivity

- Start with initial r_0 and ℓ_0 columns for POD and DEIM projection matrices.
- Iteratively determine number of additional columns to be added/deleted.

$$\begin{aligned} r^{i+1} &= r^i + \Delta r, \text{ with } \Delta r = \pm 1 + \log_{10} \left(\frac{\Delta_{POD}}{\text{tolerance}} \right) \\ \ell^{i+1} &= \ell^i + \Delta \ell, \text{ with } \Delta \ell = \pm 1 + \log_{10} \left(\frac{\Delta_{DEIM}}{\text{tolerance}} \right) \end{aligned}$$

- OFFLINE: Snapshot collection and generation of *conservative* POD and DEIM projection matrices.
- ONLINE: Adaptive process to determine an optimal POD and DEIM basis combination, from the offline matrices, respecting the required output error tolerance.

To avoid the stagnation of the iteration process (visiting same POD-DEIM basis combination more than once, or cycling repeatedly between a certain set of combinations), we incorporate a simple mechanism.

Idea:

- Store the POD/DEIM basis visited already in an array. Check if any combination is revisited, at each iteration. If so, add/subtract by one and recheck.
- Store the sign of the update at each iteration (± 1) in an array. At each iteration, determine the direction of the process as the sign of the majority of the past updates.

Results

Burgers' Equation

Figure 1: Adaptivity for Burgers' equation

n (FOM)	r_0	ℓ_0	r (ROM)	ℓ
1000	4	8	37	38

Table 1: Data for adaptive algorithm for Burgers' equation

Fluidized Bed Crystallizer (FBC)

Figure 2: Adaptivity for the FBC - (a),(b) show increasing the basis adaptively from a small r_0 and ℓ_0 (c),(d) show decreasing the basis adaptively from a large r_0 and ℓ_0

Mode	n (FOM)	r_0	ℓ_0	r (ROM)	ℓ
Increasing	18400	8	12	23	26
Decreasing	18400	37	38	11	12

Table 2: Data for adaptive algorithm for the FBC

Outlook

- Extend the adaptivity idea for parametric systems through the POD-Greedy algorithm.

References

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