The purpose of PREIM (Progressive RB-EIM) [1] is to reduce the offline costs of nonlinear parabolic reduced order models with accurate RB approximations in the online stage. The key idea is a progressive enrichment of both the EIM approximation and the RB space, in contrast to the standard approach where the EIM approximation and the RB space are built separately. PREIM uses high-fidelity computations whenever available and RB computations otherwise. Another key feature of reduced order models with accurate RB approximations in the online stage. The key idea is a progressive enhancement of the EIM approximation and the RB space, in contrast to the standard approach where the EIM approximation and the RB space are built separately.

PREIM diminishes the offline expenses in the nonlinear RB method applied to unsteady nonlinear PDEs, as long as the computation of high-fidelity trajectories is the dominant part of the offline cost.

PREIM has two goals:

- produce a set of RB functions \( \{\mathcal{W}_{m}\}_{m \in \mathbb{N}} \);
- produce a rank-approximation of the nonlinearities (5) in the form

\[
\mathcal{E}(\mu, k, \lambda) = \sum_{n} \mathcal{E}_{n}(\mu, k, \lambda),
\]

three main steps (the detailed algorithm appears in [1]):

1. select a pair

\[
(\alpha_{m}, \lambda_{m}) \in \text{argmax}_{(\alpha_{m}, \lambda_{m}) \in \mathbb{R}^{+}} \left\{ \Gamma(\mu, \alpha_{m}^{\prime}, \lambda_{m}^{\prime}) - \sum_{n=1}^{m} \mathcal{E}_{n}(\mu, \alpha_{n}^{\prime}, \lambda_{n}^{\prime}) \right\}.
\]

2. compute \( \mathcal{F}(\mu, k, \lambda) = (\alpha_{m}^{\prime}, \lambda_{m}^{\prime}) \) and update the pair

\[
(\alpha_{m+1}, \lambda_{m+1}) \in \text{argmax}_{(\alpha_{m+1}, \lambda_{m+1}) \in \mathbb{R}^{+}} \left\{ \Gamma(\mu, \alpha_{m+1}^{\prime}, \lambda_{m+1}^{\prime}) - \sum_{n=1}^{m+1} \mathcal{E}_{n}(\mu, \alpha_{n}^{\prime}, \lambda_{n}^{\prime}) \right\}.
\]

3. update the reduced basis

\[
\Sigma \leftarrow \text{POD}(\Sigma, \epsilon_{\text{basis}})
\]

with \( \Sigma \leftarrow (\mathcal{E}_{m})_{1 \leq m \leq \text{dim}X_{d}} \) and \( \text{POD} \leftarrow (u - \sum_{n=1}^{m} \mathcal{E}_{n})_{1 \leq m \leq \text{dim}X_{d}} \)

and three accuracy criteria:

- \( \epsilon_{\text{basis}} \) is the truncation threshold for the POD-based RB construction.
- \( \epsilon_{\text{approx}} \) is threshold for the approximation of the nonlinearity.
- \( \epsilon_{\text{cost}} \) is threshold for the RB approximation.