PREIM for nonlinear parabolic problems

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The purpose of PREIM (Progressive RB-EIM) [1] is to **reduce the offline costs** of nonlinear parabolic reduced order models with accurate RB approximations in the online stage. The key idea is a **progressive enrichment** of both the EIM approximation and the RB space, in contrast to the standard approach where the EIM approximation and the RB space are built separately. PREIM uses high-fidelity computations whenever available and RB computations otherwise. Another key feature of each PREIM iteration is to select twice the parameter in a greedy fashion, the second selection being made after computing the high-fidelity solution for the firstly selected value of the parameter.

Model problem

For $\mu \in \mathcal{P}^{\text{tr}}$, we consider the nonlinear heat transfer equation

$$\begin{cases} \frac{\partial u_\mu}{\partial t} - \nabla \cdot ((\kappa_0 + \Gamma(\mu, u_\mu)) \nabla u_\mu) = f, & \text{in } [0, T] \times \Omega, \\ -(\kappa_0 + \Gamma(\mu, u_\mu)) \frac{\partial u_\mu}{\partial n} = \phi_e, & \text{on } [0, T] \times \partial\Omega, \\ u_\mu(t=0, \cdot) = u_0(\cdot), & \text{in } \Omega, \end{cases} \quad (1)$$

with the standard uniform ellipticity assumption $\beta_1 \leq \kappa_0 + \Gamma(\mu, z) \leq \beta_2$ and $0 < \beta_1 < \beta_2 < \infty$.

Standard RB-EIM approach

For the RB approximation $\hat{u}_\mu^k = \sum_{n=1}^N \hat{u}_{\mu,n}^k \theta_n$, the space-time discretized PDE (1) in reduced formulation reads

$$(\mathbf{M} + \Delta t^k \mathbf{A}_0) \hat{\mathbf{u}}_\mu^k = \Delta t^k \mathbf{f}^k + \mathbf{M} \hat{\mathbf{u}}_\mu^{k-1} - \Delta t^k \mathbf{g}(\hat{\mathbf{u}}_\mu^{k-1}),$$

with

$$\mathbf{g}(\hat{\mathbf{u}}_\mu^{k-1}) = \left(\sum_{n=1}^N \hat{u}_{\mu,n}^{k-1} \int_\Omega \Gamma\left(\mu, \sum_{n'=1}^N \hat{u}_{\mu,n'}^{k-1} \theta_{n'}\right) \nabla \theta_n \cdot \nabla \theta_p \right)_{1 \leq p \leq N} \quad (2)$$

We introduce $\gamma(\mu, k, x) := \Gamma(\mu, u_\mu^k(x))$ and seek a separated approximation of γ in the form

$$\gamma_M(\mu, k, x) := \sum_{j=1}^M \phi_{\mu,j}^k q_j(x), \quad (3)$$

In the standard RB-EIM, the EIM approximation (3) and the RB space $X_N = \text{span}\{(\theta_n)_{1 \leq n \leq N}\}$ are built separately.

PREIM

We introduce

$$\tilde{u}_\mu^{m,k} := \begin{cases} u_\mu^k & \text{if } \mu \in \mathcal{P}_m^{\text{HF}}, \\ \hat{u}_\mu^{m,k} & \text{otherwise,} \end{cases} \quad (4)$$

and the nonlinear function at iteration m

$$\tilde{\gamma}^m(\mu, k, x) := \Gamma(\mu, \tilde{u}_\mu^{m,k}(x)). \quad (5)$$

PREIM has **two goals**:

- produce a set of RB functions $(\theta_n^m)_{1 \leq n \leq N^m}$;
- produce a rank- m approximation of the nonlinearity (5) in the form

$$\tilde{\gamma}_m^m(\mu, k, x) := \sum_{j=1}^m (\tilde{\phi}^m)_{\mu,j}^k \tilde{q}_j(x).$$

three main ingredients:

- $\mathcal{P}_m^{\text{HF}} \subset \mathcal{P}^{\text{tr}}$: Set that collects the previously computed HF trajectories
- $\mathcal{S}_m$: Set of interpolation points $(\tilde{x}_i)_{1 \leq i \leq m}$
- $\mathcal{Q}_m$: Set of interpolation functions $(\tilde{q}_j)_{1 \leq j \leq m}$

three main steps (the detailed algorithm appears in [1]):

- select a pair

$$(\mu_m, k_m) \in \underset{(\mu', k') \in \mathcal{P}^{\text{tr}} \times \mathbb{K}^{\text{tr}}}{\operatorname{argmax}} \|\Gamma(\mu', \tilde{u}_\mu^{k'}(\cdot)) - \tilde{\gamma}_{[\mathcal{P}_m^{\text{HF}}, \mathcal{S}_{m-1}, \mathcal{Q}_{m-1}]}^{m-1}(\mu', k', \cdot)\|_{\ell^\infty(\Omega^{\text{tr}})}.$$

- compute $\mathcal{S}_{\text{out}} = (u_\mu^k)_{k \in \mathbb{K}^{\text{tr}}}$ and update the pair

$$(\bar{\mu}_m, \bar{k}_m) \in \underset{(\mu', k') \in \mathcal{P}_{\text{out}}^{\text{HF}} \times \mathbb{K}^{\text{tr}}}{\operatorname{argmax}} \|\Gamma(\mu', u_\mu^{k'}(\cdot)) - \tilde{\gamma}_{[\mathcal{P}_m^{\text{HF}}, \mathcal{S}_{m-1}, \mathcal{Q}_{m-1}]}^{m-1}(\mu', k', \cdot)\|_{\ell^\infty(\Omega^{\text{tr}})}.$$

- update the reduced basis

$$\Xi := \text{POD}(\tilde{\mathcal{S}}, \mathcal{E}_{\text{POD}})$$

with $\Theta := (\theta_n^m)_{1 \leq n \leq N^m}$ and $\tilde{\mathcal{S}}_{\text{out}} := (u - \Pi_{\text{span}(\Theta)} u)_{u \in \mathcal{S}_{\text{out}}}$

and **three accuracy criteria**:

- $\mathcal{E}_{\text{POD}}$ is the truncation threshold for the POD-based RB construction.
- $\mathcal{E}_{\text{EIM}}$ is threshold for the approximation of the nonlinearity.
- $\mathcal{E}_{\text{RB}}$ is threshold for the RB approximation.

Numerical results

Numerical results were obtained for a nonlinearity on the solution

$$\Gamma(\mu, u) := \sin\left(\frac{2\pi\mu}{20} \left(\frac{u - u_0}{u_m - u_0}\right)^2\right)$$

and for a nonlinearity depending on partial derivatives

$$\Gamma(\mu, u) := \sin\left(\omega\mu \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)\right) \quad (6)$$

We focus on (6) with $\omega = 6.25 \cdot 10^{-3}$, $u_0 = 293$ K (20 °C), $\kappa_0 = 1$ m²·K⁻²·s⁻¹ and $\phi_e = 3$ K·m·s⁻¹.

- Space discretization: mesh containing $\mathcal{N} = 1429$ nodes.

- Time discretization:

- Discrete times nodes $\mathbb{K}^{\text{tr}} = \{1, \dots, 50\}$,
- Constant time step $\Delta t^k = 0.05$ s.

- Parametrization

- Interval $\mathcal{P} = [1, 40]$
- Training set $\mathcal{P}^{\text{tr}} = \{1, \dots, 40\}$.

Figure 1: RB approximation error $\|u_\mu - \hat{u}_\mu\|_{\ell^2(I^N; H^1(\Omega^N))}$ for $\mathcal{E}_{\text{POD}} = 5 \cdot 10^{-2}$ and $\mathcal{E}_{\text{EIM}} = 10^{-3}$

Figure 2: RB approximation error $\|u_\mu - \hat{u}_\mu\|_{\ell^2(I^N; H^1(\Omega^N))}$ for $\mathcal{E}_{\text{POD}} = 5 \cdot 10^{-2}$ and $\mathcal{E}_{\text{EIM}} = 10^{-3}$

Figure 3: RB approximation error $\|u_\mu - \hat{u}_\mu\|_{\ell^2(I^N; H^1(\Omega^N))}$ for $\mathcal{E}_{\text{POD}} = 2.5 \cdot 10^{-2}$ and $\mathcal{E}_{\text{EIM}} = 10^{-4}$

Figure 4: RB approximation error $\|u_\mu - \hat{u}_\mu\|_{\ell^2(I^N; H^1(\Omega^N))}$ for $\mathcal{E}_{\text{POD}} = 2.5 \cdot 10^{-2}$ and $\mathcal{E}_{\text{EIM}} = 10^{-4}$

Figure 5: EIM approximation errors.

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\bar{\mu}$	21	21	21	8	21	21	21	8	9	21	9	21	9	9	9	6	21
μ	21	8	21	8	21	21	21	8	9	21	9	7	6	9	9	5	4
k	2	5	3	2	50	4	49	3	4	10	50	25	49	5	10	4	6

Figure 6: Selected parameters and time nodes in PREIM for $\mathcal{E}_{\text{EIM}} = 10^{-1}$. The gray cells correspond to a new parameter selection and, therefore, to a new HF computation.

Conclusions

PREIM diminishes the offline expenses in the nonlinear RB method applied to unsteady nonlinear PDEs, as long as the computation of high-fidelity trajectories is the dominant part of the offline cost.

References

- [1] Benaceur, A., Ehrlicher, V., Ern, A., Meunier, S.: A progressive reduced basis/empirical interpolation method for nonlinear parabolic problems (2018). URL <https://hal.archives-ouvertes.fr/hal-01599304>