Linearization schemes for Hermite matrix polynomials

Heike Fassbender, Nikta Shayanfar

Objective

Root-finding of a polynomial $p(t) \in \mathbb{P}_n$ expressed in the Hermite basis, by solving a generalized eigenvalue problem.

Generalized eigenvalue problem

Given matrices $A, B \in \mathbb{C}^{n \times n}$, find all $\lambda \in \mathbb{C}$ such that for some nonzero vector $v \in \mathbb{C}^{n \times 1}$: $Av = \lambda Bv$.

- $\lambda \in \mathbb{C}$: *eigenvalue*
- nonzero $v \in \mathbb{C}^{n \times 1}$: *eigenvector*
- the pair (A, B) : *matrix pencil*

Companion pencil

A matrix pencil (C_0, C_1) in $\mathbb{C}^{n \times n}$ is required such that $\det(tC_1 - C_0) = 0$ for all t with $p(t) = 0$.

Hermite interpolation problem

Assumptions:

- Given distinct interpolation nodes $\tau_0, \dots, \tau_k \in \mathbb{C}$
- s_i function derivatives $p_{i,0}, p_{i,1}, \dots, p_{i,s_i-1}$, given at the node τ_i
- total number of functional values at all nodes:

$$\sum_{i=0}^k s_i = n + 1.$$

Goal:

Construct a polynomial $p(t) \in \mathbb{P}_n$ such that

$$p^{(j)}(\tau_i) = p_{i,j}, \quad i = 0, \dots, k, j = 0, \dots, s_i - 1.$$

Modified barycentric Hermite interpolant

$$p(t) = \omega(t) \sum_{i=0}^k \sum_{j=0}^{s_i-1} \frac{m_{i,j} w_i}{(t - \tau_i)^{s_i-j}},$$

where the barycentric weights: $w_i := \prod_{\substack{l=0, \\ l \neq i}}^n \frac{1}{(\tau_i - \tau_l)^{s_l}}$, $i = 0, \dots, k$,
 $\omega(t) = \prod_{i=0}^k (t - \tau_i)^{s_i}$, and

$$m_{i,j} := \sum_{k=0}^j \frac{p_{i,k}}{k!} \mathcal{I}_{j-k}, \quad i = 0, \dots, k, j = 0, \dots, s_i - 1,$$

where $\mathcal{I}_i, i = 0, \dots, j$ are computed in [2].

Hermite companion pencil of size $n + 2$

Let $p(t) \in \mathbb{P}_n$ presents the Hermite interpolating polynomial. Define the companion pencil $C_0^{\{n+2\}}, C_1^{\{n+2\}} \in \mathbb{C}^{(n+2) \times (n+2)}$:

$$C_0^{\{n+2\}} = \begin{bmatrix} J_{s_0}(\tau_0) & & & M_0 \\ & J_{s_1}(\tau_1) & & M_1 \\ & & \ddots & \vdots \\ & & & J_{s_k}(\tau_k) & M_k \\ -W_0 & -W_1 & \dots & -W_k & 0 \end{bmatrix}, C_1^{\{n+2\}} = \begin{bmatrix} I_{s_0} & & & \\ & I_{s_1} & & \\ & & \ddots & \\ & & & I_{s_k} \\ & & & & 0 \end{bmatrix},$$

where

$$J_{s_i}(\tau_i) = \begin{bmatrix} \tau_i & 1 & & \\ & \tau_i & \ddots & \\ & & \ddots & 1 \\ & & & \tau_i \end{bmatrix} \in \mathbb{C}^{s_i \times s_i}, \quad i = 0, \dots, k,$$

$$W_i = [w_i \ 0 \ \dots \ 0] \in \mathbb{C}^{1 \times s_i}, \quad i = 0, \dots, k,$$

$$M_i = [m_{i,s_i-1} \ \dots \ m_{i,1} \ m_{i,0}]^T \in \mathbb{C}^{s_i \times 1}, \quad i = 0, \dots, k.$$

The companion pencil satisfies $p(t) = \det(tC_1^{\{n+2\}} - C_0^{\{n+2\}}) [1]$.

Hermite companion pencil of size $n + 1$

Let $p(t) \in \mathbb{P}_n$ presents the Hermite interpolating polynomial. Define the companion pencil $C_0^{\{n+1\}}, C_1^{\{n+1\}} \in \mathbb{C}^{(n+1) \times (n+1)}$:

$$C_0^{\{n+1\}} = \begin{bmatrix} J_{s_0}(\tau_0) & & & \tau_k \tilde{M}_0 \\ & J_{s_1}(\tau_1) & & \tau_k \tilde{M}_1 \\ & & \ddots & \vdots \\ & & & J_{s_{k-1}}(\tau_{k-1}) & \tau_k \tilde{M}_{k-1} \\ -\tilde{W}_0 & -\tilde{W}_1 & \dots & -\tilde{W}_{k-1} & \tilde{J}_k \end{bmatrix}, C_1^{\{n+1\}} = \begin{bmatrix} I_{s_0} & & & \tilde{M}_0 \\ & I_{s_1} & & \tilde{M}_1 \\ & & \ddots & \vdots \\ & & & I_{s_{k-1}} & \tilde{M}_{k-1} \\ & & & & \tilde{I}_k \end{bmatrix},$$

where

$$\tilde{J}_k := \begin{bmatrix} J_{s_{k-1}}(\tau_k) & \tau_k \tilde{M}_k \\ \tilde{W}_k & 0 \end{bmatrix}, \tilde{I}_k := \begin{bmatrix} I_{s_{k-1}} & \tilde{M}_k \\ 0_{1 \times s_{k-1}} & 0 \end{bmatrix} \in \mathbb{C}^{s_k \times s_k},$$

for $i = 0, \dots, k - 1$:

$$\tilde{M}_i = -[0_{s_i \times s_{k-1}} \ m_{k,0}^{-1} M_i] \in \mathbb{C}^{s_i \times s_k}, \quad \tilde{W}_i = \begin{bmatrix} 0_{s_{k-1} \times s_i} \\ W_i \end{bmatrix} \in \mathbb{C}^{s_k \times s_i},$$

and

$$\tilde{M}_k := -m_{k,0}^{-1} [m_{k,s_k-1} \ \dots \ m_{k,2} \ m_{k,1}]^T \in \mathbb{C}^{s_k-1 \times 1},$$

$$\tilde{W}_k := [w_k \ 0 \ \dots \ 0] \in \mathbb{C}^{1 \times s_k-1}.$$

The companion pencil satisfies $p(t) = m_{k,0} \det(tC_1^{\{n+1\}} - C_0^{\{n+1\}})$.

References:

- [1] H. Fassbender, N. Shayanfar, *Linearization of matrix polynomials in Hermite basis*, unpublished.
- [2] B. Sadiq, D. Viswanath, *Barycentric Hermite interpolation*, SIAM J. Sci. Comput. 35 (2013) 1254–1270.