Sampling-free parametric model reduction of systems with structured parameter variation

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Introduction

Consider a parametric LTI dynamical systems represented as

$$E\dot{x}(t;p) = A(p)x(t;p) + Bu(t;p),$$
$$y(t;p) = Cx(t;p),$$

where $E, A(p) \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{l\times n}$, $[\,E, A(p), B, C\,]$.

- $x(t;p) \in \mathbb{R}^n$ denotes the state variable
- $u(t) \in \mathbb{R}^m$ and $y(t;p) \in \mathbb{R}^l$ represent the inputs and outputs of the system, resp.
- $A(p)$ depends on $k \ll n$ parameters $p = (p_1, p_2, \ldots, p_k)$ such that we may write

$$A(p) = A_0 + U\ \mathrm{diag}(p_1, p_2, \ldots, p_k)V^T,$$

where $p_i \geq 0$ for $i = 1, \ldots, k$ and $U, V \in \mathbb{R}^{n\times k}$ are fixed.

Full-order transfer function

$$\mathbf{H}(s;p) = C(sE - A(p))^{-1}B.$$

Approximating the full-order transfer function

We would like to produce a reduced order model that retains the structure of parametric dependence and offers uniformly high fidelity across the full parameter range. For the full-order transfer function:

$$\mathbf{H}(s;p) = C\left(\widehat{A}(s) + U\ \mathrm{diag}(p_1, p_2, \ldots, p_k)V^T\right)^{-1}B, \quad \text{where} \quad \widehat{A}(s) = sE - A_0.$$

Most parametric model reduction methods require parametric sampling. In order to remove this need which, requires identifying particular parameter values of interest, we use the Sherman-Morrison-Woodbury formula:

$$\mathbf{H}(s;p) = \mathbf{H}_1(s) - \mathbf{H}_2(s)D(p)(I_k + D(p)\mathbf{H}_3(s)D(p))^{-1}D(p)\mathbf{H}_4(s), \tag{1}$$

where parameters are encoded in diagonal matrix $D(p) = \mathrm{diag}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_k})$ and

$$\mathbf{H}_1(s) = C\widehat{A}(s)^{-1}B, \qquad \mathbf{H}_2(s) = C\widehat{A}(s)^{-1}U,$$
$$\mathbf{H}_3(s) = V^T\widehat{A}(s)^{-1}U, \qquad \mathbf{H}_4(s) = V^T\widehat{A}(s)^{-1}B.$$

We construct a parameterized reduced order model by reducing four subsystems

$$[\,E, A_0, B, C\,],\ [\,E, A_0, U, V^T\,],\ [\,E, A_0, U, C\,],\ \text{and}\ [\,E, A_0, B, V^T\,],$$

which do not depend on parameters. Additionally the first subsystem does not depend neither on matrices U, V.

Parameterized reduced order model

Reduced model based on reduction of subsystems systems

- For four underlaying subsystems calculate reduced systems using well-established model reduction techniques for non-parametric systems

$$[\,E, A_0, B, C\,] \to G_1^r(s),$$
$$[\,E, A_0, U, V^T\,] \to G_2^r(s),$$
$$[\,E, A_0, U, C\,] \to G_3^r(s),$$
$$[\,E, A_0, B, V^T\,] \to G_4^r(s),$$

e. g. using balanced truncation or IRKA approach. These reductions do not depend on parameters!

- For any given parameter $p = (p_1, p_2, \ldots, p_k)$ calculate approximated system by

$$G^r(s;p) = G_1^r(s) - G_2^r(s)D(p)(I_k + D(p)G_3^r(s)D(p))^{-1}D(p)G_4^r(s)$$

- Use this reduced parametric system in optimization and/or inverse problems.

Reduced model based on vector fitting approach

- For the predetermined points in the complex plane $\xi_1, \ldots, \xi_N$ calculate

$$\mathbf{H}_1(\xi_i),\ \mathbf{H}_2(\xi_i),\ \mathbf{H}_3(\xi_i),\ \mathbf{H}_4(\xi_i) \quad \text{for} \quad i = 1, \ldots, N.$$

These samples do not depend on parameters!

- For any given parameter $p = (p_1, p_2, \ldots, p_k)$ calculate $\mathbf{H}(\xi_i;p)$ for $i = 1, \ldots, N$ using formula (1).
- Based on $\mathbf{H}(\xi_1;p), \ldots, \mathbf{H}(\xi_N;p)$ calculate reduced system with transfer function $\mathbf{H}^r(s;p)$ using vector fitting approach.
- Use this reduced parametric system in optimization and/or inverse problems.

The overall process is well suited for computationally efficient parameter optimization and the study of important system properties.

Applications to damping optimization

We consider a vibrational system described by

$$M\ddot{q}(t;p) + (C_{int} + C_{ext}(p))\dot{q}(t;p) + Kq(t;p) = Ew(t),$$
$$z(t;p) = Hq(t;p), \tag{2}$$

- mass matrix, M, and stiffness matrix, K, are real, symmetric positive-definite matrices of order n,
- $q(t;p)$ is a vector of displacements and rotations,
- $z(t;p)$ and $w(t)$ represent, respectively, the outputs and the inputs (typically viewed as potentially disruptive) of the system,
- damping in the structure is modeled by

$$C_{int} + C_{ext}(p),$$

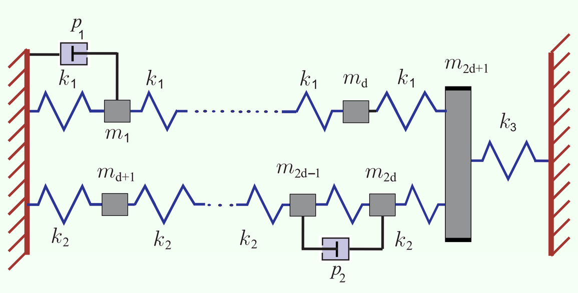
C_{int} represents internal damping,
$C_{ext}(p) = U\,\mathrm{diag}\,(p_1, p_2, \ldots, p_k)U^T$ represents external damping with gains $p_1, p_2, \ldots, p_k$ and $U \in \mathbb{R}^{n\times k}$ determines the placement and geometry of the external dampers.

Main problem: to determine the best damping matrix that is able to minimize influence of the disturbances, w, on the output of the system z.

Optimization criterion: $\mathcal{H}_2$ system norm.

Example

Consider a mass oscillator with $2d + 1$ masses and $2d + 3$ springs.



The mathematical model is given by (2) with

$$K = \begin{bmatrix} K_{11} & & -\kappa_1 \\ & K_{22} & -\kappa_2 \\ -\kappa_1^T & -\kappa_2^T & k_1 + k_2 + k_3 \end{bmatrix},\quad K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},\quad \kappa_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_i \end{bmatrix},$$

for $i = 1, 2$ and $M = \mathrm{diag}\,(m_1, m_2, \ldots, m_n)$.
$d = 900 \Rightarrow n = 1801$, with $m_{1801} = 1000$ and

$$m_i = \begin{cases} 1000 - \frac{i}{2}, & i = 1, \ldots, 450, \\ i + 325, & i = 451, \ldots, 900, \\ 1300 - \frac{i}{4}, & i = 901, \ldots, n. \end{cases}$$

The stiffness values are given by

$$k_1 = 500, k_2 = 200, k_3 = 300.$$

The primary excitation are 5 disturbances applied to the 4 masses closest to the left-hand side and one mass closest to the right-hand side of oscillator.
We are interested in 2 displacements, i. e.

$$z(t;p) = \left[q_{400}(t;p)\ q_{1300}(t;p)\right]^T.$$

Internal damping is a small multiple of critical damping

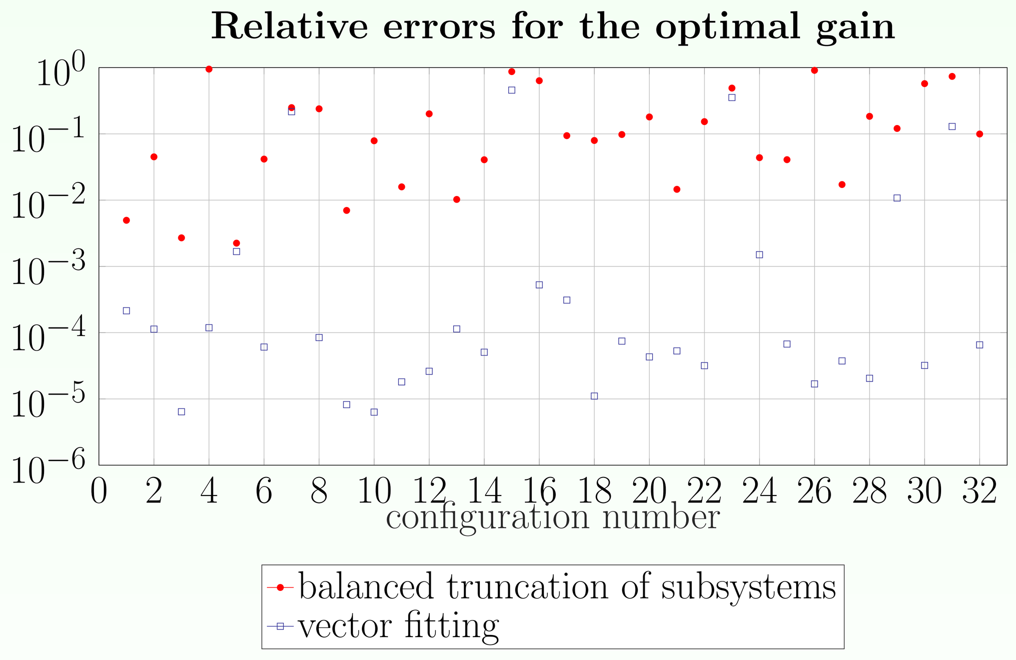
$$C_{int} = 0.04 \cdot M^{1/2}\left(M^{-1/2}KM^{-1/2}\right)^{1/2} M^{1/2}.$$

We consider four dampers with gains p_1, p_2, p_3 and p_4 where geometry of positions is given by

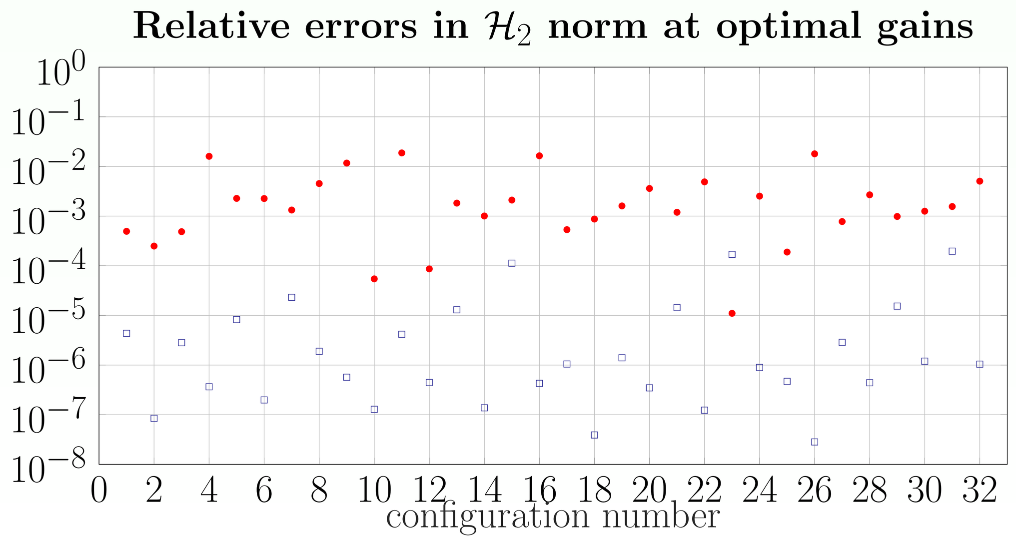
$$U = \left[e_{j_1} - e_{j_1+10},\ \ e_{j_2},\ \ e_{j_3},\ \ e_{j_3} - e_{j_3+100}\right],$$

with $j_1 \in \{100, 300, 500, 700\}$, $j_2 \in \{150, 350, 550, 750\}$, $j_3 \in \{1400, 1700\} \Rightarrow 32$ different damping configurations.

Relative errors for the optimal gain



Relative errors in $\mathcal{H}_2$ norm at optimal gains



Gains were optimized with starting point $p^0 = (100, 100, 100, 100)$ using the full-order model and using proposed reduced systems. Relative errors in optimal values and relative errors $\mathcal{H}_2$ norm at optimal values values are presented.

In the approach based on **balanced truncation of subsystems:**

- in all damping configurations, subsystems were reduced to dimensions 500, 400, 300, 350, respectively.

In the approach based on **vector fitting approach**

- initial points ξ_i, $i = 1, \ldots, N$, for $N = 400$ were calculated by using dominant poles,
- order of approximation was 240.

In considered parameterized reduced order models there was no need for parameter sampling.

Time ratio

In average case for one optimization of parameters, new approach **was faster**:

- $\approx$ 8 times, with usage of reduced model based on balanced truncation of subsystems,
- $\approx$ 75 times, with usage of reduced model based on vector fitting approach.

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