Article title: A proof of four color conjecture (4CC)

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A proof of four-color conjecture (4CC)

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Abstract: found out a critical corresponding relationship between a δ -wheel of any maximal simple plane graph of n vertices (n -MSPG) and its corresponding δ -cycle of a corresponding $n-1$ -MSPG of the n -MSPG, and relied on the inductive way, while with the help of KempeChain (KC), this article gives a brand-new logical proof for the famous and more than 100 years old four-color conjecture (4CC).

Keywords: n -MSPG; $n-1$ -MSPG, corresponding; δ -wheel; δ -cycle; KempeChain (KC); at least 3-vertex colorable; 4-vertex colorable

4CC was brought out by Francis Guthrie in 1852 [1]-[3]: if four colors were sufficient to color the plane district map of England or any other plane map? A.Kempe once wrote a paper in *nature* in 1879 to declare the conjecture had been proved, later the detailed proof was published in *Amer. J. Math.* 2 in 1879 [4]. But P.J.Heawood found a mistake of the proof of A.Kempe, which cannot be changed, while he proved five-color theorem of plane graph [5]. The conjecture is famous because until now there has no admitted logical proof of it appeared except a 1200 hours computer proof had made by K. Appel, W. Haken, and J.Koch until 1976[6]- [8].

In Graph Theory, the 4CC can be described: if any simple plane graph of n

vertices (n -SPG) is 4-vertex colorable? [1]- [3]

We have known a maximal n -SPG (n -MSPG) is an n -SPG which has no edge can be added.

By the way, we have to note there is no a polygon but triangles in an n -MSPG; if it is not so, edges can be added on the polygon to make it to be composed only by triangles; the whole border of an n -MSPG is also a triangle, when the whole outside of an n -MSPG was looked as the inside of the biggest triangle.

There is an obvious relationship between an n -SPG and an n -MSPG: every n -SPG is a sub graph of a certain n -MSPG of all n -MSPGs; so we have a basic conclusion: if every n -MSPG can be 4-vertex colorable, then every n -SPG can be 4-vertex colorable.

How to prove every n -MSPG can be 4-vertex colorable?

Proof is by induction on the vertices of the n -MSPG and by a critical corresponding relationship between any n -MSPG and a corresponding $n-1$ -MSPG of it.

The sole 4-MSPG is obviously 4-vertex colorable; it is easy to verify every 5-MSPG and every 6-MSPG are 4-vertex colorable etc; so we can assume every $n-1$ -MSPG has been 4-vertex colored; we need to prove that every n -MSPG can be 4-vertex colorable.

It is obvious that every n -MSPG must have at least one smallest δ -wheel, δ notes the minimum number of edges stick to a common vertex of the

n -MSPG; there is a not obvious but most critical corresponding relationship between any n -MSPG and a corresponding $n-1$ -MSPG of it: for any n -MSPG there is a corresponding $n-1$ -MSPG which has the same structure with the n -MSPG at any place except only at a corresponding δ -cycle while in the n -MSPG the δ -cycle has correspondingly turned into the corresponding δ -wheel. See Figure1.

Figure 1 an example of a 16-MSPG with its δ -wheel and its corresponding 15-MSPG with its δ -cycle

If the δ -cycle can be at least 3-vertex colorable in the already been 4-vertex colored $n-1$ -MSPG while remain the whole $n-1$ -MSPG still 4-vertex colorable, then the corresponding δ -wheel can be 4-vertex colorable while the corresponding whole n -MSPG can be 4-vertex colorable.

A famous theorem has proved $\delta \leq 5$. [1]-[3]

If $\delta = 3$, the 3-cycle has already been at least 3-vertex colorable in the already 4-vertex colored $n-1$ -MSPG.

If $\delta = 4$ and the 4-cycle has not been at least 3-vertex colored but 4-vertex colored in the already 4-vertex colored $n-1$ -MSPG, We can remove the only edge in the 4-cycle; see Figure2; because it is obvious that a complete red-yellow-red-...-yellow or briefly $r-y-r-\dots-y$ Kempe Chain (ryKC) between the red(r) and the yellow(y) while a complete green-blue-green-...-blue or briefly $g-b-g-\dots-b$ Kempe Chain (gbKC) between the green(g) and the blue(b) cannot exist together at the outside of the 4-cycle; for example the complete ryKC is not exist, we can start from the r along with its all ryKCs to exchange r and y ; finally the r has changed into the y while the y is unchanged; so the 4-cycle is at least 3-vertex colorable in the already 4-vertex colored $n-1$ -MSPG while remain the whole $n-1$ -MSPG still 4-vertex colorable.

Figure 2 a 4-wheel and its corresponding 4-cycle & related KC

If $\delta = 5$ and the 5-cycle has already 4-vertex colored in the already 4-vertex colored n-1-MSPG, We remove the just two edges in the 5-cycle; see Figure3; generally, there must be one color is just between two same color; such as the r is between the two same y; as there is no a complete rgKC between the r and the g we can reduce the three colors of r, g, b to two colors of r, g, b by using the same way mentioned above; that means the 4-vertex colored 5-cycle can be at least 3-vertex colorable in the already 4-vertex colored n-1-MSPG while remain the whole n-1-MSPG still 4-vertex colorable; as both the complete rgKC and the complete rbKC exist we can change the y in the y-r-g-r...g cycle to b while we can change the y in the y-r-b-r...b cycle to g; that also means the 4-vertex colored 5-cycle can be at least 3-vertex colorable in the already 4-vertex colored n-1-MSPG while

remain the whole $n-1$ -MSPG still 4-vertex colorable.

Figure 3 a 5-wheel and its corresponding 5-cycle & related KC

In summarize, every n -MSPG can be 4-vertex colorable.

So every n -SPG can be 4-vertex colorable.

In some sense this proof gives a way of how to color an n -SPG to make it 4-vertex colorable: if you want to 4-vertex color an n -SPG, you need to 4-vertex color an n -MSPG firstly; if you want to 4-vertex color an n -MSPG, you can to 4-vertex color an $n-1$ -MSPG firstly; if you want to 4-vertex color an $n-1$ -MSPG, you can to 4-vertex color an $n-2$ -MSPG firstly; and so on; until to a 4-MSPG. Even more, we need remark you again and again that every time from a 4-vertex colorable $j-1$ -MSPG to a 4-vertex colorable j -MSPG you need adjust the j -MSPG's color once more by using the way mentioned above. As the same of n , $j \geq 5$, j belongs to N , while N is the set of nature numbers.

By the way, This proof gives a corresponding relationship between the set of all $n-1$ -MSPGs and the set of all n -MSPGs; that is not mean one $n-1$ -MSPG has only one corresponding n -MSPG because the making up of the δ -cycle or precisely the Hamiltonian- δ -cycle[1]-[3] is flexible in the $n-1$ -MSPG; in fact, one $n-1$ -MSPG has many corresponding n -MSPGs, on the contrary, one n -MSPG has at least one corresponding $n-1$ -MSPG; there is a little of regret that we still don't know the accurate number relationship between the set of $n-1$ -MSPGs and the set of n -MSPGs.

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